Real Analysis Notes

Rational Numbers and Bounds

If $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd} \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

provided that $\frac{c}{d} \neq \frac{0}{1}$.

Note. Strictly speaking, we need to show that these operations are **well-defined** or that they don't depend on the choice of representatives from the equivalence classes.

Definition. Suppose S is an ordered set, and $E \subseteq S$. If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say E is **bounded above** and we call β an **upper bound**. The terms **bounded below** and **lower bound** are defined similarly.

Definition. Suppose S is an ordered set, $E \subseteq S$, and E is bounded above. Suppose there exists $\alpha \in S$ such that α is an upper bound for E and if $\gamma < \alpha$, then γ is not an upper bound for E, then α is the **least upper bound** of E or the **supremum** of E, and we write $\alpha = \sup E$. The **greatest lower bound** and **infimum** ($\inf E$) are defined similarly.

Example. Consider the set $\{r \in \mathbb{Q} : r^2 < 2\}$, which has no supremum in $\mathbb{Q}$.

Definition. An ordered set S has the **least-upper-bound property** if the following is true: if $E \subseteq S$, E is not empty, and E is bounded above, then $\sup E$ exists in S.

Proposition. If an ordered set has the least-upper-bound property, then it also has the greatest-lower-bound property.

Definition. There exists an ordered field $\mathbb{R}$ (called the **real numbers**) which has the least-upper-bound property, and it contains an isomorphic copy of $\mathbb{Q}$.

Note. Finite ordered fields do not exist. Consider $0 \leq 1 \leq 1+1 \leq \ldots$ which can't be a finite chain.

Dedekind Cuts

1. Define the elements of $\mathbb{R}$ as subsets of $\mathbb{Q}$ called **cuts**, where a cut is a subset α of $\mathbb{Q}$ such that
   (a) α is a nonempty proper subset of $\mathbb{Q}$ ($\alpha \neq \varnothing$ and $\alpha \neq \mathbb{Q}$).
   (b) If $p \in \alpha$, $q \in \mathbb{Q}$, and $q < p$, then $q \in \alpha$.

(c) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$ (can't be in the set and be an upper bound).

2. Define an order on $\mathbb{R}$ where $\alpha < \beta$ if and only if α is a proper subset of β.

3. Show that the ordered set $\mathbb{R}$ has the least-upper-bound property. To do this, suppose A is a nonempty subset of $\mathbb{R}$ that is bounded above. Let γ be the union of all $\alpha \in A$. Then show $\gamma \in \mathbb{R}$ and $\gamma = \sup A$.

4. For $\alpha, \beta \in \mathbb{R}$, define the sum $\alpha + \beta$ to be the set of all sums $r + s$ where $r \in \alpha$ and $s \in \beta$. Define $0^* = \{t \in \mathbb{Q} : t < 0\}$ then show axioms for addition in fields hold for $\mathbb{R}$, and that 0^* is the additive identity.

5. Show that if $\alpha, \beta, \gamma \in \mathbb{R}$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$. This is part of showing that $\mathbb{R}$ is an ordered field.

6. For $\alpha, \beta \in \mathbb{R}$, where $\alpha > 0^*$ and $\beta > 0^*$, define the product $\alpha\beta$ to be $\{p \in \mathbb{Q} : q \leq rs,\ r \in \alpha,\ s \in \beta,\ r > 0,\ s > 0\}$. Note that $\alpha\beta > 0^*$ if $\alpha > 0^*$ and $\beta > 0^*$, which is part of showing that $\mathbb{R}$ is an ordered field.

7. Extend the definition of multiplication to all of $\mathbb{R}$ by setting, for all $\alpha, \beta \in \mathbb{R}$, $\alpha 0^* = 0^* \alpha = 0^*$ and
$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \alpha < 0^*, \beta < 0^* \\ -[(-\alpha)(\beta)] & \alpha < 0^*, \beta > 0^* \\ -[(\alpha)(-\beta)] & \alpha > 0^*, \beta < 0^* \end{cases}$$
then prove the distributive law.

8. Associate to each $r \in \mathbb{Q}$ the real number $r^* = \{t \in \mathbb{Q} : t < r\}$ and let $\mathbb{Q}^* = \{r^* : r \in \mathbb{Q}\}$. These are the rational cuts in $\mathbb{R}$.

9. Show that $\mathbb{Q}$ is isomorphic to $\mathbb{Q}^*$ as ordered fields.

Properties of Real Numbers

Theorem. Any two ordered fields with the least upper-bound-property are isomorphic.

Theorem. If $x, y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. This is called the **Archimedean property** of $\mathbb{R}$.

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$ and suppose the Archimedean property is false. Then y would be an upper bound of A. But then A would have a least upper bound. Say $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound. Thus, $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. But then $\alpha < (m+1)x$, which contradicts the fact that α is an upper bound of A. Thus, the Archimedean property must be true.

Theorem. If $x, y \in \mathbb{R}$ and $x < y$, then there exists $p \in \mathbb{Q}$ such that $x < p < y$. We say that $\mathbb{Q}$ is **dense** in $\mathbb{R}$.

Theorem. For every positive real number x and every positive integer n, there is exactly one positive real number y such that $y^n = x$.

Proof. There is at most one since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$. Let $E = \{t \in \mathbb{R} : t > 0, t^n < x\}$. Then E is nonempty since $t = \frac{x}{1+x} \implies 0 < t < 1 \implies t^n < t < x \implies t \in E$. We also know E is bounded above since $t > 1 + x \implies t^n > t > x \implies t \notin E$ and t is an upper bound. Define $y = \sup E$. We can then show that $y^n < x$ and $y^n > x$ each lead to contradictions.

Question. Given a real number in decimal form, what is its associated Dedekind cut?

Cardinality of Sets

Definition. Let A and B be sets. If there is a bijection from A to B, then we say A and B have the same **cardinality** (or 'size') and write $A \sim B$. We also write $|A| = |B|$ where $|A|$ denotes the cardinality of A.

Definition. Let $\mathbb{N}$ denote the natural numbers $\{1, 2, 3, \ldots\}$, also denoted $\mathbb{Z}^+$. For $n \in \mathbb{N}$, let $J_n = \{1, 2, \ldots, n\}$ and $J_0 = \varnothing$. For any set A,

1. A is **finite** if $A \sim J_n$ for some $n \in \mathbb{N} \cup \{0\}$.
2. A is **infinite** if it is not finite.
3. A is **countable** if $A \sim \mathbb{N}$.
4. A is **uncountable** if A is neither finite nor countable.
5. A is **at most countable** if A is finite or countable.

Note. We can put an order on the cardinalities where $|A| \leq |B|$ if and only if there exists an injection from A to B.

Proposition. Every infinite subset of a countable set is countable.

Proposition. Let $\{E_n\}$ where $n \in \mathbb{Z}^+$ be a sequence of countable sets. If $S = \cup_{n=1}^{\infty} E_n$, then S is countable.

Proof. Let the elements of E_i be as follows

$$\begin{aligned}
E_1 &= \{x_{11}, x_{12}, x_{13}, \ldots\} \\
E_2 &= \{x_{21}, x_{22}, x_{23}, \ldots\} \\
&\cdots \\
E_i &= \{x_{i1}, x_{i2}, x_{i3}, \ldots\} \\
&\cdots
\end{aligned}$$

We can traverse these elements diagonally to get $S = \{x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, \ldots\}$. Since S is at most countable and $E_1 \subseteq S$ is countable, we have that S is countable.

Proposition. Let A be a countable set and B_n be the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_k \in A$ for $k = 1, 2, \ldots, n$. Then B_n is countable for all $n \in \mathbb{N}$.

Theorem. Let A be the set of all sequences of 0's and 1's. Then A is uncountable.

Proof. Let $E = \{e_1, e_2, \ldots\}$ be a countable subset of A. For each e_i, we analyze its ith digit. We then construct $e \in A$ such that the ith digit of e is the opposite of the ith digit of e_i. For example, if we have

$$\begin{aligned} e_1 &= (\boxed{0}, 1, 0, 1, 1, 1, 0, 1, \ldots) \\ e_2 &= (1, \boxed{1}, 0, 1, 0, 1, 1, 0, \ldots) \\ e_3 &= (0, 0, \boxed{1}, 1, 0, 0, 1, 1, \ldots) \\ e_4 &= (1, 0, 1, \boxed{0}, 1, 0, 1, 1, \ldots) \\ &\cdots \end{aligned}$$

Then $e = (1, 0, 0, 1, \ldots)$. Since $e \notin E$ but $e \in A$, every countable subset of A is a proper subset of A. Thus, A is uncountable.

Metric Spaces

Definition

A set X, whose elements we will call **points**, is a **metric space** if there is a function $d : X \times X \to \mathbb{R}$ such that $\forall p, q \in X$

1. $d(p, q) > 0$ if $p \neq q$, and $d(p, p) = 0$.
2. $d(p, q) = d(q, p)$.
3. $d(p, q) \leq d(p, r) + d(r, q)$, $\forall r \in X$ (triangle inequality).

Definition. The number $d(p, q)$ is the **distance** from p to q, and d is a **metric**.

Note. $\mathbb{R}^k$ is a metric space with the usual metric $d(x, y) = |x - y|$, $x, y \in \mathbb{R}^k$.

Proposition. Every subset Y of a metric space X is also a metric space where we restrict the metric of X to points in Y.

Open and Closed Sets

Definition. Let X be a metric space, $p \in X$, and $E \subseteq X$.

1. Let $r \in \mathbb{R}^+$. The **neighborhood** of p with **radius** r is the set $N_r(p) = \{q \in X : d(p, q) < r\}$.
2. The point p is a **limit point** of E if every neighborhood of p contains a point $q \in E$ and $q \neq p$.
3. If $p \in E$ and p is not a limit point of E, then p is an **isolated point** of E.

4. E is **closed** if every limit point of E is in E.

5. If $p \in E$ and there is an $r \in \mathbb{R}^+$ such that $N_r(p) \subseteq E$, then p is an **interior point** of E.

6. E is **open** if every point of E is an interior point of E.

7. The **complement** of E in X is the set $E^c = \{x \in X : x \notin E\}$.

8. E is **perfect** if it is closed and if every point of E is a limit point of E.

9. E is **bounded** if there exists a number $M \in \mathbb{R}^+$ and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.

10. E is **dense** in X if every point of X is in E or a limit of E (or both).

Proposition. Every neighborhood is an open set.

Proof. Consider the neighborhood $E = N_r(p)$, and let $q \in E$. Then $r - d(p, q) \in \mathbb{R}^+$. For all points s such that $d(q, s) < r - d(p, q)$, we have $d(p, s) \leq d(p, q) + d(q, s) < d(p, q) + (r - d(p, q)) = r$, implying $s \in N_r(p)$. Thus, q is an interior point of E, and the result follows.

Proposition. If p is a limit point of E, then every neighborhood of p contains infinitely many points of E.

Proposition. Let X be a metric space and suppose $E \subseteq X$. The set E is open in X if and only if its complement is closed.

Proof. We first prove the forward direction then the backward direction.

($\Rightarrow$) Suppose E is open. If x is a limit point of E^c, then every neighborhood of x contains a point of E^c. In this case, x can't be an interior point of E, and because E is open, $x \in E^c$. Thus, E^c is closed.

($\Leftarrow$) Now suppose E^c is closed. If $x \in E$ then $x \notin E^c$ and is thus not a limit point of E^c. In this case, there is a neighborhood $N(x)$ such that $N(x) \cap E^c = \varnothing$, implying that $N(x) \subseteq E$. Thus, x is an interior point and E is open.

Theorem. Consider the following statements regarding unions and intersections of open and closed sets.

1. For any collection $\{G_\alpha\}$ of open sets, $\cup_\alpha G_\alpha$ is open.

2. For any collection $\{F_\alpha\}$ of closed sets, $\cap_\alpha F_\alpha$ is closed.

3. For any finite collection $G_1, G_2, \ldots, G_n$ of open sets, $\cap_{i=1}^n G_i$ is open.

4. For any finite collection $F_1, F_2, \ldots, F_n$ of closed sets, $\cup_{i=1}^n F_i$ is closed.

Definition. Let X be a metric space. If $E \subseteq X$, let E' be the set of limit points of E. The **closure** of E is the set $\overline{E} = E \cup E'$.

Proposition. If X is a metric space and $E \subseteq X$, then

1. $\overline{E}$ is closed.

2. $E = \overline{E}$ if and only if E is closed.

3. $\overline{E} \subseteq F$ for every closed subset F of X such that $E \subseteq F$.

Note. $\overline{E}$ is the smallest closed set that contains E.

Compactness

Definition. Let X be a metric space, and let $E \subseteq X$. An **open cover** of E is a collection $\{G_\alpha\}$ of open subsets of X such that $E \subseteq \cup_\alpha G_\alpha$.

Definition. If $\{G_\alpha\}$ is an open cover of E, then a subset of $\{G_\alpha\}$ that is also an open cover of E is called a **subcover** of $\{G_\alpha\}$.

Definition

A subset K of a metric space is **compact** if every open cover of K contains a finite subcover.

Proposition. Every finite set in a metric space is compact.

Proposition. Compact subsets are closed.

Proof. Let K be a compact subset of a metric space X. We will show that K is closed by showing that K^c is open. Suppose $p \in K^c$. We will show that K^c is open by showing that p is an interior point of K^c. For each $q \in K$, let V_q and W_q be neighborhoods of p and q, of radius less than half the distance between p and q. Since K is compact, there are finitely many points, say $q_1, \ldots, q_n$ in K such that if $W = W_{q_1} \cup W_{q_2} \cup \ldots \cup W_{q_n}$, then $K \subseteq W$. If $V = V_{q_1} \cap V_{q_2} \cap \ldots \cap V_{q_n}$, then V is a neighborhood of p that does not intersect W, which covers K. Thus, $V \subseteq K^c$, and p is therefore an interior point of K^c.

Proposition. Suppose $K \subseteq X \subseteq Y$. Then K is compact relative to X if and only if K is compact relative to Y.

Proposition. Closed subsets of compact sets are compact.

Proposition. If F is closed and K is compact, then $F \cap K$ is compact.

Proposition. If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof. Suppose no point of K is a limit point of E. Then each point $q \in K$ would have a neighborhood V_q that contains at most one point of E (namely q, if $q \in E$). But then no finite subset of $\{V_q\}$ can cover E, and the same is true for K because $E \subseteq K$. This contradicts the fact that K is compact. Thus, the theorem follows.

Theorem

Let E be a subset of $\mathbb{R}^k$ (viewed as a metric space with the usual metric). The following are equivalent.

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E.

Note. The Heine-Borel Theorem is "(1) if and only if (2)" for $\mathbb{R}^k$.

Note. For all metric spaces, "(2) if and only if (3)" holds.

Perfect Sets

Definition. Let X be a metric space, and E be a subset of X. We say E is **perfect** if

1. E is closed and
2. Every point of E is a limit point of E.

Proposition. If P is a nonempty perfect set in $\mathbb{R}^k$, then P is uncountable.

Proof. Since P has limit points, we know P is infinite. Suppose P is countable and define the points of P by $x_1, x_2, \ldots$. Let V_1 be any neighborhood of x_1. If V_1 has radius r, note that $\overline{V_1} = \{y \in \mathbb{R}^k : |y - x_1| \leq r\}$. We will use V_1 to recursively construct a sequence $\{V_n\}$ of neighborhoods as follows. Suppose V_n has been constructed so that $V_n \cap P$ is not empty. Since every point of P is a limit point of P, there is a neighborhood V_{n+1} such that $\overline{V_{n+1}} \subseteq V_n$, $x_n \notin \overline{V_{n+1}}$, and $V_{n+1} \cap P \neq \{\}$. By the last condition, our recursive construction can proceed to give us a sequence $\{V_n\}$ of neighborhoods. Let $K_n = \overline{V_n} \cap P$. Since $\overline{V_n}$ is closed and bounded, $\overline{V_n}$ is compact and K_n is compact. Since $x_n \notin \overline{V_{n+1}}$, no point of P is contained in $\cap_{n=1}^{\infty} K_n$. But since $K_n \subseteq P$, this implies that $\cap_{n=1}^{\infty} K_n$ is empty. But each K_n is not empty by the fact that $V_{n+1} \cap P \neq \{\}$ and $K_n \supseteq K_{n+1}$. But this contradicts the corollary that if $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supseteq K_{n+1}$, then $\cap_{n=1}^{\infty} K_n$ is not empty. The theorem follows.

Proposition. Let $a, b \in \mathbb{R}$ and $a < b$. Then the interval $[a, b]$ is uncountable. Also, $\mathbb{R}$ is uncountable.

Note. There are, however, perfect sets in $\mathbb{R}$ that contain no intervals.

Example. Let E_0 be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$ and let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Removing the middle thirds from these intervals yields $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continuing gives us a sequence $\{E_n\}$ of compact sets such that

1. $E_1 \supseteq E_2 \supseteq E_3 \supseteq \ldots$ and
2. E_n is the union of 2^n disjoint intervals, each of length $\frac{1}{3^n}$.

Then the set $P = \cap_{n=1}^{\infty} E_n$ is called the **Cantor set**. Note that P is compact. Also, it is not empty. P contains no intervals and P is perfect.

Convergence and Limits

Definition

A sequence $\{p_n\}$ in a metric space X is said to **converge** if there is a point $p \in X$ such that for every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \epsilon$. In this case, we say $\{p_n\}$ **converges to** p, or that p is the **limit** of $\{p_n\}$, and we write

$$\lim_{n\to\infty} p_n = p$$

Definition. We say $\{p_n\}$ is **bounded** if the set of all p_n is bounded.

Definition. If $\{p_n\}$ does not converge, then it **diverges**.

Proposition. Let $\{p_n\}$ be a sequence in a metric space X.

1. $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.
2. If $\{p_n\}$ converges to p and p', then $p = p'$.
3. If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
4. If $E \subseteq X$ and p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $\lim_n \to \infty p_n = p$.

Proposition. Suppose $\{s_n\}$ and $\{t_n\}$ are complex sequences, and $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Then,

1. $\lim_{n\to\infty}(s_n + t_n) = s + t$.
2. $\lim_{n\to\infty}(c + s_n) = c + s$ and $\lim_{n\to\infty} cs_n = cs$.
3. $\lim_{n\to\infty} s_n t_n = st$.
4. $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided that $s_n \neq 0$ and $s \neq 0$.

Proof Idea. The key insight for (3) is that $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$.

Proposition. If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Proof. Let $\epsilon > 0$ and let N be a positive integer grater than $\frac{1}{\epsilon^{1/p}}$. If $n \geq N$, then $\frac{1}{n^p} \leq \frac{1}{N^p} < \frac{1}{(1/\epsilon^{1/p})^p} = \epsilon$. It follows that $\frac{1}{n^p} \to 0$.

Proposition. If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

Proof. Case $p > 1$. Put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$ and by the Binomial Theorem, $1 + nx_n \leq (1 + x_n)^n = p$. Thus, $0 < x_n \leq \frac{p-1}{n}$. Hence $x_n \to 0$ so $\sqrt[n]{p} \to 1$.

Case $p = 1$. This is trivial.

Case $0 < p < 1$. Consider the sequence $\{\frac{1}{\sqrt[n]{p}}\}$. By the first case, $\frac{1}{\sqrt[n]{p}} = \sqrt[n]{\frac{1}{p}} \to 1$. Thus, $\sqrt[n]{p} \to 1$.

Proposition. If $n > 0$, then $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. Let $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and by the Binomial Theorem, $n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2$. Thus when $n \geq 2$, $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$. But $\sqrt{\frac{2}{n-1}} \to 0$, thus $x_n \to 0$ so $\sqrt[n]{n} \to 1$.

Proposition. If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.

Proof. Let k be a positive integer such that $k > \alpha$. When $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} > \frac{n}{2} \cdot \frac{n}{2} \cdot \dots \cdot \frac{n}{2} \cdot \frac{p^k}{k!} = \frac{n^k}{2^k} \frac{p^k}{k!}$$

Thus, when $n > 2k$,

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{n^\alpha}{n^k/2^k \cdot p^k/k!} = \frac{2^k k!}{p^k} n^{\alpha-k}$$

Since $\alpha - k < 0$, $n^{\alpha-k} = \frac{1}{n^{k-\alpha}} \to 0$. Therefore, $\frac{n^\alpha}{(1+p)^n} \to 0$.

Proposition. If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof. Let $p = \frac{1}{|x|} - 1$. Then $p > 0$ and setting $\alpha = 0$ in the identity $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ yields

$$0 = \lim_{n\to\infty} \frac{1}{(1+p)^n} = \lim_{n\to\infty} \frac{1}{(1/|x|)^n} = \lim_{n\to\infty} |x|^n$$

It follows that $x^n \to 0$.

Cauchy Sequences

Definition

A sequence $\{p_n\}$ in a metric space X is a **Cauchy sequence** if for every $\epsilon > 0$, there is an integer N such that $d(p_n, p_m) < \epsilon$ if $m, n \geq N$.

Proposition. In any metric space X, every convergent sequence is a Cauchy sequence.

Proposition. If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence, then $\{p_n\}$ converges to some point in X.

Proposition. In $\mathbb{R}^k$, every Cauchy sequence converges.

Definition. A metric space is **complete** if every Cauchy sequence converges.

Proposition. All compact metric spaces and all Euclidean spaces are complete metric spaces.

Note. How might we 'complete' the rationals? We can think of $\mathbb{R}$ as equivalences classes of Cauchy sequnces such that $\{p_n\} \sim \{q_n\}$ if and only if $d(p_n, q_n) \to 0$.

Definition

Given a complex sequence $\{a_n\}$, we can create a new sequence $\{s_n\}$ where $s_n = \sum_{k=1}^{n} a_k$. We call $\{s_n\}$ an (infinite) **series** and is often denoted $\sum_{n=1}^{\infty} a_n$. The numbers s_n are the **partial sums** of the series. Also, if $s_n \to s$, then we will write

$$\sum_{n=1}^{\infty} a_n = s$$

Note. Sometimes we will consider series of the form $s_0, s_1, s_2, \ldots$ and write $\sum_{n=0}^{\infty} a_n$. We might also just write $\sum a_n$.

Proposition. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if for every $\epsilon > 0$, there is an integer N such that $|\sum_{k=n}^{m} a_k| < \epsilon$ whenever $m \geq n \geq N$.

Proof. Note that $\mathbb{C}$ is essentially $\mathbb{R}^2$ and thus $\{s_n\}$ converges if and only if it is a Cauchy sequence. Furthermore if $m \geq n-1$, then $d(s_m, s_{n-1}) = |s_m - s_{n-1}| = |\sum_{k=n}^{m} a_k|$.

Tests for Convergence

Proposition. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof. Take $m = n$ in the previous theorem, which gives us $|a_n| < \epsilon$ when $n \geq N$.

Note. It is possible for $a_n \to 0$ and have $\sum a_n$ be divergent. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Proof Idea. Consider the following.

$$\underbrace{1 + 1/2}_{>1/2} + \underbrace{1/3 + 1/4}_{>1/2} + \underbrace{1/5 + 1/6 + 1/7 + 1/8}_{>1/2} + \underbrace{1/9 + \ldots + 1/16}_{>1/2} + \ldots$$

Theorem (Comparison Test). Let N_0 be a fixed integer. If $|a_n| \leq c_n$ for $n \geq N_0$, and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof. Given $\epsilon > 0$, we know there is some $N > N_0$ such that $m \geq n \geq N$ implies $\sum_{k=n}^{m} c_k < \epsilon$. Thus, $|\sum_{k=n}^{m} a_k| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \epsilon$ and the first part follows. Also, the second part follows from the first part because if $\sum a_n$ converges, then so must $\sum d_n$.

Note. To use the Comparison Test, we need to know a series of nonnegative real numbers whose convergence or divergence is known.

Proposition. If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x = 1$, the series diverges.

Proof. The key insight is that if $x \neq 1$, we let $s_n = \sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$ and the result follows by letting $n \to \infty$. Note that when $x = 1$, the series clearly diverges.

Theorem (Cauchy Condensation Test). Suppose $a_1 \geq a_2 \geq \ldots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$ converges.

Proof. A series of nonnegative real terms converges if and only if its partial sums form a bounded sequence. Thus, it suffices to consider the boundedness of the partial sums $s_n = a_1 + a_2 + \ldots + a_n$ and $t_k = a_1 + 2a_2 + \ldots + 2^k a_{2^k}$. For $n < 2^k$, $s_n \leq t_k$ because

$$\begin{aligned} s_n &\leq (a_1) + (a_2 + a_3) + \ldots + (a_{2^k} + \ldots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \ldots + 2^k a_{2^k} \\ &= t_k \end{aligned}$$

For $n > 2^k$, $2s_n \geq t_k$ because

$$\begin{aligned} s_n &\geq (a_1) + (a_2) + (a_3 + a_4) + \ldots + (a_{2^{k-1}+1} + \ldots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \ldots + 2^{k-1}a_{2^k} \\ &= \frac{1}{2}t_k \end{aligned}$$

It follows that the sequences $\{s_n\}$ and $\{t_n\}$ are either both bounded or both unbounded and the theorem follows.

Proposition (p-Series Test)**.** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof. If $p \geq 0$, then the terms in the sequence don't converge to 0 so $\sum \frac{1}{n^p}$ diverges. If $p > 0$, then by the previous theorem, we can consider the series

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} (\frac{1}{2^{p-1}})^k$$

But this is a geometric series which will converge if $0 \leq \frac{1}{2^{p-1}} < 1$ and will diverge if $\frac{1}{2^{p-1}} \geq 1$ so the original series will converge if $p > 1$ and diverge if $p \leq 1$.

Definition. Let $\{s_n\}$ be a sequence of real numbers and let $\{s_{n_k}\}$ be some subsequence that converges to some x. Then x is a **subsequential limit** of $\{s_n\}$.

Definition. Let $\{s_n\}$ be a sequence of real numbers and let E be the set of all subsequential limits. We define

$$\limsup_{n\to\infty} s_n = \sup E \qquad \liminf_{n\to\infty} s_n = \inf E$$

Theorem (The Root Test)**.** Given $\sum a_n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

1. If $\alpha < 1$, then $\sum a_n$ converges.
2. If $\alpha > 1$, then $\sum a_n$ diverges.
3. If $\alpha = 1$, then the test gives no info.

Proof. If $\alpha < 1$, then choose β such that $\alpha < \beta < 1$ and an integer N such that $\sqrt[n]{|a_n|} < \beta$ for $n \geq N$. Thus if $n \geq N$, we have $|a_n| < \beta^n$. Since $0 < \beta < 1$, $\sum \beta^n$ converges because it is a geometric series and by the comparison test, $\sum a_n$ converges.

If $\alpha > 1$, then there is a sequence $\{n_k\}$ such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$. Thus $|a_n| > 1$ for infinitely many n, so it must be the case that $a_n \not\to 0$ and therefore $\sum a_n$ can't converge.

Note that $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ both have $\alpha = 1$ but they diverge and converge, respectively.

Theorem (The Ratio Test). Suppose $a_n \neq 0$ for all n. The series $\sum a_n$

1. Converges if $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$.
2. Diverges if $|\frac{a_{n+1}}{a_n}| \geq 1$ for all $n \geq n_0$ where n_0 is some fixed integer.

Proof. If $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then choose some $\beta < 1$ and an integer N such that $|\frac{a_{n+1}}{a_n}| < \beta$ for all $n \geq N$. We then have for all $n \geq N$, $|a_n| < (|a_N|\beta^{-N})\beta^n$. Since $0 < \beta < 1$, $\sum \beta^n$ converges. Thus by the comparison test, $\sum a_n$ converges.

If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, then $a_n \not\to 0$, so $\sum a_n$ diverges.

Note. The Ratio Test is not useful for $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$.

Example. Consider the series $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \ldots$. In this case, $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{8}$, $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$, so the Ratio Test does not apply. But $\lim_{n\to\infty} \sqrt[n]{a_n} = 2$ so the series converges by the Root Test.

Properties of Convergence

Definition. Let $z \in \mathbb{C}$ and let $\{c_n\}$ be a sequence of complex numbers. The series $\sum c_n z^n$ is a (complex) **power series**.

Proposition. Given the power series $\sum c_n z^n$, let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and let $R = \frac{1}{\alpha}$ (where $\alpha = 0 \implies R = +\infty$ and $\alpha = +\infty \implies R = 0$). Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Proof. Let $a_n = c_n z^n$ and apply the Root Test. We see that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = |z|\alpha = \frac{|z|}{R}$$

Note that if $\frac{|z|}{R} > 1$ then $|z| > R$ and the series diverges. Similarly, if $\frac{|z|}{R} < 1$, then $|z| < R$ and the series converges.

Proposition. Given two sequences $\{a_n\}$ and $\{b_n\}$, let $A_n = \sum_{k=0}^{n} a_k$ be the n^{th} partial sum of $\sum a_n$ for $n \geq 0$ and $A_{-1} = 0$. If $0 \leq p \leq q$, then

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Proof. We have

$$\begin{aligned}\sum_{n=p}^{q} a_n b_n &= \sum_{n=p}^{q} (A_n - A_{n-1}) b_n \\ &= \sum_{n=p}^{q} A_n b_n - \sum_{n=p}^{q} A_{n-1} b_n \\ &= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\end{aligned}$$

Note. This is a "partial summation formula". We will use it to understand series of the form $\sum a_n b_n$.

Proposition. Suppose the partial sums A_n of $\sum a_n$ form a bounded sequence and $b_0 \geq b_1 \geq b_2 \geq \dots$ and $\lim_{n\to\infty} b_n = 0$. Then $\sum a_n b_n$ converges.

Proof. Choose M such that $|A_n| \leq M$ for all n. Given $\epsilon > 0$, there is an integer N such that $b_N < \frac{\epsilon}{2M}$. If $N \leq p \leq q$, then

$$\begin{aligned}\left|\sum_{n=p}^{q} a_n b_n\right| &= \left|\sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p\right| \\ &\leq M \left|\sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q\right| + M\left|b_p\right| \\ &= M(2b_p) \\ &\leq 2Mb_p \\ &\leq 2Mb_N \\ &< \epsilon\end{aligned}$$

The convergence of $\sum a_n b_n$ follows from the Cauchy Criterion.

Proposition. Suppose $|c_1| \geq |c_2| \geq |c_3| \geq \dots$, $c_{2m-1} \geq 0$, $c_{2m} \leq 0$, and $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n$ converges.

Proof. Apply the previous proposition with $a_n = (-1)^{n+1}$ and $b_n = |c_n|$.

Definition. An **alternating series** is a series for which $c_{2m-1} \geq 0$ and $c_{2m} \leq 0$.

Definition. The series $\sum a_n$ is said to **converge absolutely** if the series $\sum |a_n|$ converges.

Proposition. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.

Proof. Convergence follows from the inequality $|\sum_{k=n}^{m} a_k| \leq \sum_{k=n}^{m} |a_k|$ and the Cauchy Criterion.

Definition. If $\sum a_n$ converges but not absolutely, then we say it converges **conditionally**.

Note. The Comparison Test, Root Test, and Ratio Test are really tests for absolute convergence. They don't give any information about conditionally convergent series.

Definition. The **sum** of two series $\sum a_n$ and $\sum b_n$ is the series $\sum d_n$ where $d_n = a_n + b_n$ for all n. The **product** of $\sum a_n$ and $\sum b_n$ is the series $\sum c_n$ where $c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$.

Proposition. If $\sum a_n = A$ and $\sum b_n = B$, then $\sum (a_n + b_n) = A + B$ and $\sum c a_n = cA$.

Note. It is possible to take the product of two convergent series and yield a series that does not converge.

Proposition. Suppose $\sum a_n$ converges absolutely, $\sum a_n = A$, $\sum b_n = B$, and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $\sum c_n = AB$.

Proposition. If $\sum a_n = A$, $\sum b_n = B$, $\sum c_n = C$, and $c_n = a_0 b_n + \ldots + a_n b_0$, then $AB = C$.

Definition. Let $\{k_n\}$ for $n = 1, 2, 3, \ldots$ be a sequence in which every positive integer appears once and only once. If $a'_k = a_{k_n}$ then we say $\sum a'_n$ is a **rearrangement** of $\sum a_n$.

Example. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges to some nonzero real number, say A. If we could rearrange the terms without changing the limit A, we would see that

$$\begin{aligned}
A &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \\
&= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \ldots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \ldots \\
&= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots\right) \\
&= \frac{1}{2}A
\end{aligned}$$

But then $A = 0$, which is a contradiction of $A \neq 0$.

Proposition. Let $\sum a_n$ be a real series that converges but not absolutely. If $-\infty \leq \alpha \leq \beta \leq +\infty$, then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that $\liminf_{n\to\infty} s'_n = \alpha$ and $\limsup_{n\to\infty} s'_n = \beta$.

Proposition. If $\sum a_n$ converges absolutely, then every rearrangement of $\sum a_n$ converges and they all converge to the same number.

Continuity

Definition

Let X and Y be metric space with metrics d_X and d_Y, respectively. If p is a limit point of X, and $f : X \to Y$, then we write
$$\lim_{x \to p} f(x) = q$$
if there is a point $q \in Y$ with the following property.

For every $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all $x \in X$ for which $d_X(x, p) < \delta$.

In this case, we say that "the limit of $f(x)$ as x approaches p is a".

Definition

Suppose X and Y are metric spaces, $p \in X$, and $f : X \to Y$. We say f is **continuous at p** if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all $x \in X$ such that $d_X(x, p) < \delta$.

Definition

If f is continuous at every point of X, then we say f is **continuous on X**, or simply that f is **continuous**.

Example. Every function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.

Theorem. Suppose p is a limit point of X and $f : X \to Y$. Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Theorem. Suppose $f : X \to Y$. The map f is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

Proof. We first prove the forward direction then the backward direction.

($\Rightarrow$) Suppose f is continuous on X and V is an open set of Y. We will show every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. Suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\epsilon > 0$ such that $y \in V$ if $d_X(f(p), y) < \epsilon$. Since f is continuous on p, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ if $d_X(x, p) < \delta$. Thus, the neighborhood $N_\delta(p)$ is contained in $f^{-1}(V)$. It follows that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. Thus $f^{-1}(V)$ is open as desired.

($\Leftarrow$) Suppose $f^{-1}(V)$ is open in X for every open set V in Y. Fix $p \in X$ and $\epsilon > 0$ and let $V = N_\epsilon(f(p))$. Then V is open, so $f^{-1}(V)$ is open. Hence there exists $\delta > 0$ such that $N_\delta(p) \subseteq f^{-1}(V)$. In other words, there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \epsilon$ for all $x \in X$ for which $d_X(x, p) < \delta$. Thus f is continuous at p. It follows that f is continuous on X.

Proposition. Suppose $f : X \to Y$. The map f is continuous on X if and only if $f^{-1}(K)$ is closed for every closed set K in Y.

Proposition. Suppose X and Y are metric spaces and that X is compact. If $f : X \to Y$ is continuous, then $f(X)$ is compact.

Proof. Let $\{V_\alpha\}$ be an open covering of $f(X)$. Since f is continuous, $\{f^{-1}(V_\alpha)\}$ is an open covering of X. Since X is compact, there are finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X \subseteq f^{-1}(V_{\alpha_1}) \cup \ldots \cup f^{-1}(V_{\alpha_n})$. Thus, $f(X) \subseteq V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$ implying that $f(X)$ is compact.

Example. There is no continuous map from $[0,1]$ to $\mathbb{R}$.

Proposition. If f is a continuous map of a compact metric space X into $\mathbb{R}^k$, then $f(X)$ is closed and bounded.

Proposition. Suppose f is a continuous real function on a compact space X, $M = \sup f(x)$, and $m = \inf f(x)$. Then there are points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Proposition. Let X and Y be metric spaces and suppose $f : X \to Y$ is continuous and bijective. If X is compact, then $f^{-1} : Y \to X$ is continuous.

Proof. It is enough to show that for every open set V in X, $(f^{-1})^{-1}(V) = f(V)$ is open in Y. Let V be an open set in X. Then V^c is closed in X, thus V^c is compact in X. Hence, $f(V^c)$ is compact in Y and is therefore closed in Y. Since f is bijective, $f(V)^c = f(V^c)$. Thus, $f(V)$ is open.

Definition. Let X and Y be metric spaces and $f : X \to Y$. We say f is **uniformly continuous** on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all $p, q \in X$ for which $d_X(p, q) < \delta$.

Proposition. Let X and Y be metric spaces and suppose $f : X \to Y$ is continuous. If X is compact, then f is uniformly continuous.

Derivatives

Definition

Let f be a real-valued function on $[a,b]$. For any $x \in [a,b]$, define

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

provided that the limit exists. Then f' is called the **derivative** of f. If f' is defined on x, then f is **differentiable at x**. If f' is defined at every point $x \in E \subseteq [a,b]$, then f is **differentiable on E**.

Theorem. Let f be a function on $[a,b]$. If f is differentiable at $x \in [a,b]$, then f is continuous at x.

Proposition. If $f : [a,b] \to \mathbb{R}$, $f : [a,b] \to \mathbb{R}$, and f and g are both differentiable, then fg is differentiable.

Proof. We first note that $fg(t) - fg(x) = (f(t) - f(x))g(x) + f(t)(g(t) - g(x))$. Then we observe that

$$\lim_{t \to x} \frac{fg(t) - fg(x)}{t - x} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} g(x) + \lim_{t \to x} f(t) \frac{g(t) - g(x)}{t - x}$$

Thus, $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Note. Similar properties hold for the sum and quotient of two functions. The proof and derivation are similar.

Note. Because you know the derivative of $f(x) = c$ and $f(x) = x$, you can use the product rule, sum rule, and induction to show that the derivatives of polynomials are what you know they are. You induct by using the identity

$$a_n x^n + \ldots + a_1 x + a_0 = x(a_n x^{n-1} + \ldots + a_2 x + a_1) + a_0$$

Theorem (Generalized Mean Value Theorem)**.** If f and g are continuous real functions on a closed interval $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

Proof. Define $h : [a, b] \to \mathbb{R}$ by setting $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t)$. Then h is continuous on $[a, b]$ and differentiable on (a, b). Also, $h(a) = f(b)g(a) - f(a)g(b) = h(b)$. The theorem will follow if we show that $h'(x) = 0$ for some $x \in (a, b)$. If h is constant then we are done. If $h(t) > h(a)$ for some $t \in (a, b)$, then let x be a point in (a, b) at which h attains its maximum (by the compactness of $[a, b]$). Thus, $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a, b)$, then the same result holds by using the minimum of h.

Theorem (Mean Value Theorem)**.** If f is real and continuous on $[a, b]$ and differentiable on (a, b), then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = f'(x)(b - a)$$

Corollary. Suppose f is differentiable on (a, b).

1. $f'(x) \geq 0$ for all $x \in (a, b) \implies f$ is monotonically increasing.
2. $f'(x) = 0$ for all $x \in (a, b) \implies f$ is constant.
3. $f'(x) \leq 0$ for all $x \in (a, b) \implies f$ is monotonically decreasing.

Proposition. Suppose f is a real differentiable function on $[a, b]$. If $f'(a) < \lambda < f'(b)$, there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

Proof. Set $g(t) = f(t) - \lambda t$. Then $g'(a) = f'(a) - \lambda < 0$ so $g(t_1) < g(a)$ for some $t_1 \in (a, b)$. Also, $g'(b) = f'(b) - \lambda > 0$ so $g(t_2) < g(b)$ for some $t_2 \in (a, b)$. Thus, we know g attains its minimum on $[a, b]$ at some point $x \in (a, b)$. Thus, $g'(x) = 0$ and $f'(x) = \lambda$.

Theorem (Taylor's Theorem)**.** Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, and $f^{(n)}(t)$ exists for all $t \in (a, b)$. Let $\alpha, \beta \in [a, b]$ where $\alpha \neq \beta$ and define

$$\begin{aligned} P_{n-1}(t) &= f(\alpha) + f'(\alpha)(t - \alpha) + \frac{1}{2!}f''(\alpha)(t - \alpha)^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(\alpha)(t - \alpha)^{n-1} \\ &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k \end{aligned}$$

Then there exists a point x between α and β such that $f(\beta) = P_{n-1}(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$.

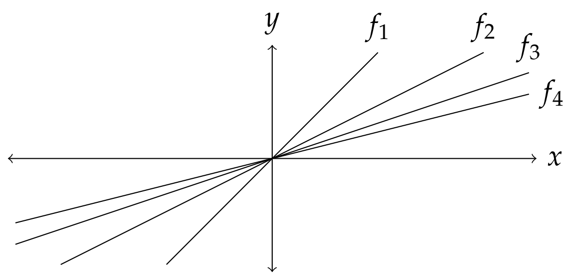
Proof. Let M be the number such that $f(\beta) = P_{n-1}(\beta) + M(\beta - \alpha)^n$. We want to show that $M = \frac{f^{(n)}(x)}{n!}$ for some x between α and β. To do so, define $g : [a, b] \to \mathbb{R}$ by setting $g(t) = f(t) - P_{n-1}(t) - M(t-\alpha)^n$. Then $g^{(n)}(t) = f^{(n)}(t) - n!M$ for all $t \in (a, b)$. Thus, we need to show that $g^{(n)}(x) = 0$ for some x between α and β. Since $P_{n-1}^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, 1, \dots, n-1$, we have $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$. Next note that $g(\beta) = 0$ so by the Mean Value Theorem, $g'(x_1) = 0$ for some x_1 between α and β. Since $g'(\alpha) = 0$ and $g'(x_1) = 0$, by the Mean Value Theorem, $g''(x_2) = 0$ for some x_2 between α and x_1. After n such steps, we have $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1}. Since x_n is also between α and β, this completes the proof.

Note. When $n = 1$, this is just the Mean Value Theorem.

Note. The polynomial P_{n-1} is a **Taylor polynomial** and we can ask questions about the sequence $P_0, P_1, P_2, \dots$ such as does it converge.

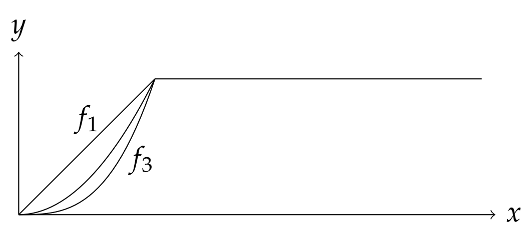
Convergence of Functions

Consider the following example of a sequence of functions $f_1, f_2, f_3, \dots$ where $f_n = \frac{x}{n}$ ($f_n : \mathbb{R} \to \mathbb{R}$). Then the picture we have is



and the functions are converging "pointwise" to the constant function $f(x) = 0$. Next consider the functions f_n defined on $[0, \infty)$ where

$$f_n = \begin{cases} x^n & x \in [0,1] \\ 1 & x \in (1, \infty) \end{cases}$$



The picture here is as shown and the limit approaches a function resembling a step. In this, each f_n is continuous, but the pointwise limit is not continuous.

Definition. A sequence of functions $\{f_n\}$ **converges uniformly** to a function f if for every $\epsilon > 0$ there is an integer N such that $m \geq N$ implies $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in E$.

Proposition (Cauchy Criterion). The sequence $\{f_n\}$ converges uniformly if and only if for every $\epsilon > 0$ there exists an integer N such that $n \geq N$ and $m \geq N$ implies $|f_n(x) - f_m(x)| \leq \epsilon$ for all $x \in E$.

Theorem. Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose $\lim_{t\to x} f_n(t) = A_n$ $(n = 1, 2, 3, \ldots)$. Then $\{A_n\}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$. In other words,

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$

Proof. Let $\epsilon > 0$. By the uniform convergence of $\{f_n\}$, there exists an N such that $n \geq N$, $m \geq N$, $t \in E$ imply $|f_n(t) - f_m(t)| \leq \epsilon$. Letting $t \to x$ yields $|A_n - A_m| \leq \epsilon$ when $n \geq N$ and $m \geq M$, showing that $\{A_n\}$ is a Cauchy sequence (of complex numbers) and therefore converges to some A. Note that for $n \in \mathbb{N}$ and $t \in E$,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$$

by the Triangle Inequality. Now choose n such that for all $t \in E$, $|f(t) - f_n(t)| \leq \frac{\epsilon}{3}$ (which we can do since $f_n \to f$ uniformly) and such that $|A_n - A| \leq \frac{\epsilon}{3}$. Then for this n, choose a neighborhood V of x such that $|f_n(t) - A_n| \leq \frac{\epsilon}{3}$ when $t \in V \cap E$, $t \neq x$. It follows that $|f(t) - A| \leq \epsilon$ when $t \in V \cap E$, $t \neq x$, which is equivalent to $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

Corollary. If $\{f_n\}$ is a sequence of continuous functions, and if $f_n \to f$ uniformly, then f is continuous.

Proof. By the previous theorem, $\lim_{t\to x} f_n(t) = A_n = f_n(x)$ and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} f_n(x) = f(x)$.

Definition. If X is a metric space, then $\mathscr{C}(X)$ denotes the set of all continuous, complex-valued, and bounded functions with domain X.

Definition. We can associate to each $f \in \mathscr{C}(X)$ its **supremum norm**

$$\|f\| = \sup_{x\in X} |f(x)|$$

Proposition. Suppose $f, g \in \mathscr{C}(X)$.

1. $\|f\| < \infty$.
2. $\|f\| = 0$ if and only if f is the zero function.
3. $\|f + g\| \leq \|f\| + \|g\|$.

Proposition. Define the distance between f and g to be $\|f - g\|$. Then $\mathscr{C}(X)$ is a metric space.

Proposition. A sequence $\{f_n\}$ converges to f in $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.

Proposition. The above metric makes $\mathscr{C}(X)$ into a complete metric space.

Proof. Suppose $\{f_n\}$ is a Cauchy sequence in $\mathscr{C}(X)$. Thus for every $\epsilon > 0$, there is an N such that $\|f_n - f_m\| < \epsilon$ whenever $n \geq N$ and $m \geq N$. Thus there is a function $f : X \to \mathbb{C}$ to which $\{f_n\}$ converges uniformly. Thus f is continuous. Also, f is bounded since there is an n such that $|f(x) - f_n(x)| < 1$ for all $x \in X$ and f_n is bounded. Hence $f \in \mathscr{C}(X)$ and since $f_n \to f$ uniformly on X, we have $\|f - f_n\| \to 0$ as $n \to \infty$.

Theorem (Stone-Weierstrass Theorem)**.** If f is a continuous complex function on $[a, b]$, then there exists a sequence of polynomials $\{P_n\}$ such that $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[a, b]$. Furthermore, if f is real, then P_n can be taken to be real.