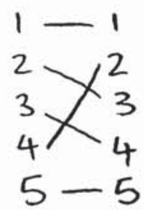
The <u>symmetric group</u> S_n is the set of bijections on $\{1, 2, \dots, n\}$ and is a group under composition. Elements of S_n are called <u>permutations</u>. Every permutation admits a cycle decomposition.

<u>Ex:</u> $\sigma \in S_5$ $\quad \sigma(1) = 1 \quad \sigma(2) = 3 \quad \sigma(3) = 4 \quad \sigma(4) = 2 \quad \sigma(5) = 5$



OR $\quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 \end{pmatrix}$ $\quad$ OR $\quad (2\,3\,4)$

(1) and (5) are called <u>fixed points</u>.

<u>Def:</u> The <u>cycle type</u> of $T \in S_n$ is $(1^{m_1}, 2^{m_2}, \dots, n^{m_n})$ where m_i is the number of cycles of length i in σ.

<u>Ex:</u> $\sigma = (2\,3\,4) \in S_5$ has cycle type $(1^2, 2^0, 3^1, 4^0, 5^0)$

<u>Def:</u> An <u>involution</u> is a permutation σ such that $\sigma^2 = (1) \in S_n$.

<u>Claim:</u> σ is an involution iff $m_i = 0$ for $i > 2$, i.e. σ only has fixed points and transpositions in its cycle decomposition.

<u>Def:</u> A <u>partition</u> of $n \in \mathbb{Z}$ is a monotonically decreasing sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \qquad \lambda_i \geq \lambda_{i+1}$$

such that

$$\sum_{i=1}^{\ell} \lambda_i = n$$

<u>Note:</u> Each partition of n determines and is determined by a cycle type in S_n. $\quad \sigma = (2\,3\,4), \ \lambda = (3, 1, 1)$

<u>Def:</u> In any group G, elements $g, h \in G$ are <u>conjugate</u> if $\exists x \in G$ s.t.

$$g = xhx^{-1}$$

<u>Remark:</u> Conjugation is an equivalence relation.

<u>Cor:</u> Conjugacy classes partition G.

<u>Ex:</u> Elements in S_n are conjugate iff they have the same cycle type.

The conjugacy class of $g \in G$ is denoted K_g.

<u>Def:</u> Let $g \in G$. The <u>centralizer</u> of g is $Z_g = \{h \in G : hgh^{-1} = g\}$.

<u>Claim:</u> There exists a bijection between the cosets of Z_g and K_g.

<u>Pf:</u> $$xZ_g = yZ_g \iff y^{-1}x \in Z_g \iff y^{-1}xg(y^{-1}x)^{-1} = g$$
$$\iff y^{-1}xgx^{-1}y = g \iff xgx^{-1} = ygy^{-1}$$

<u>Cor:</u> $|K_g| = |G|/|Z_g|$ for finite G.

<u>Notation:</u> If $g \in S_n$ has cycle type λ, we write
$$K_g = K_\lambda \quad Z_g = Z_\lambda \quad z_\lambda = |Z_\lambda|$$

<u>Prop:</u> Let $g \in S_n$ have type $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ then $|Z_g|$ depends only on λ and
$$z_\lambda = \prod_{i=1}^{n} i^{m_i} m_i!$$

<u>Def:</u> A <u>matrix representation</u> of G is a group homomorphism $G \to GL_d$

<u>Ex:</u> The <u>trivial representation</u> of G is $1_G : G \to GL_1$ defined by $1_G(g) = (1), \forall g \in G$

<u>Def:</u> d is the <u>degree</u> or <u>dimension</u> of the representation $X : G \to GL_d$.

<u>Ex:</u> The trivial representation is degree 1.

<u>Ex:</u> Let C_n be the cyclic group of order n.

<u>Def:</u> Let V be a vector space. The <u>general linear group</u> of V is
$$GL(V) = \{T : V \to V \mid T \text{ is an isomorphism}\}$$

<u>Def:</u> Let G be a group, V a vector space. Then V is a <u>G-module</u> if there exists a group homomorphism
$$\rho : G \to GL(V)$$

Def: Let G be a group and A a set. Then G acts on A if there is a map $G \times A \to A$, $(g,a) \mapsto ga$ such that (1) $ea = a$, $\forall a \in A$, (2) $g(ha) = (gh)a$, $\forall g,h \in G$. This map is called the <u>action</u> of G on A.

Note: Every group acts on itself via multiplication.

Note: S_n acts on $\{1,\dots,n\}$. $S_n \times \{1,\dots,n\} \to \{1,\dots,n\}$, $(\pi,k) \mapsto \pi(k)$

Claim: V is a G-module iff G acts on V, i.e. $G \times V \to V$, $(g,v) \mapsto gv$ such that

(1) $ev = v$
(2) $g(hv) = (gh)v$
(3) $g(av + bw) = agv + bgw$

Pf: Let $\rho: G \to GL(V)$ be a group homomorphism. Define an action $(g,v) \mapsto \rho(g)(v)$. Note $\rho(e)(v) = I(v) = v$, $\rho(gh)(v) = \rho(g)(\rho(h)(v))$ since ρ is a homomorphism, and $g(av+bw) = agv + bgw$ because $\rho(g) \in GL(V)$ is linear. Similarly, an action $(g,v) \mapsto gv$ defines a homomorphism $\rho(g)(v) = gv$.

Def: Let G act on a finite set $S = \{s_1, \dots, s_n\}$. The <u>permutation representation</u> is $\mathbb{C}S = \{c_1 s_1 + \dots + c_n s_n \mid c_i \in \mathbb{C}\}$. under formal linear combination. This is a vector space. Then $\{s_1,\dots,s_n\}$ form a basis for the vector space $\mathbb{C}S$, called the <u>standard basis</u>, and $\dim \mathbb{C}S = n$.

$\mathbb{C}S$ is a module with the action of G given by $g\left(\sum c_i s_i\right) = \sum c_i (g s_i) \in \mathbb{C}S$.

Ex: S_n acts on $\{1,\dots,n\} = S$. $\mathbb{C}S = \{c_1\vec{1} + \dots + c_n\vec{n}\}$

$$S_n \times \mathbb{C}S \to \mathbb{C}S \qquad \pi(c_1\vec{1} + \dots + c_n\vec{n}) = c_1\pi(\vec{1}) + \dots + c_n\pi(\vec{n})$$

Consider $n=3$, $\pi = (1\ 2\ 3)$. The standard basis is $\{\vec{1}, \vec{2}, \vec{3}\}$. We have $\pi(\vec{1}) = \vec{2}$, $\pi(\vec{2}) = \vec{3}$, $\pi(\vec{3}) = \vec{1}$.

$$X(\pi) = X(1\ 2\ 3) = \left([X(\vec{1})]\,[X(\vec{2})]\,[X(\vec{3})] \right) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$ permutation matrix

Def: Let G be a group. The (left) regular representation of G is given by G acting on itself, in more detail, the group algebra of G to be formal sums

$$\mathbb{C}[G] = \{c_1 g_1 + \ldots + c_n g_n \mid c_i \in \mathbb{C}\} \text{ and } G = \{g_1, \ldots, g_n\}$$

with

$$\left(\sum c_i g_i\right) + \left(\sum d_i g_i\right) = \sum (c_i + d_i) g_i \quad , \quad c\left(\sum c_i g_i\right) = \sum (c c_i) g_i ,$$

$$g\left(\sum c_i g_i\right) = \sum c_i (g g_i), \quad \left(\sum c_i g_i\right)\left(\sum d_i g_i\right) = \sum (c_i d_j)(g_i g_j)$$

Note: $\mathbb{C}[G]$ is a G-module.

Ex: What is the regular representation of C_4? $\quad C_4 = \{g, g^2, g^3, g^4 = e\}$

$$\mathbb{C}[C_4] = \{c_1 \vec{g} + c_2 \vec{g^2} + c_3 \vec{g^3} + c_4 \vec{e}\} \qquad g^3 \vec{g} = \vec{e}, \; g^3 \vec{g^2} = \vec{g}, \; g^3 \vec{g^3} = \vec{g^2}, \; g^3 \vec{e} = \vec{g^3}$$

so we have $\quad X(\vec{g^3}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Remark: The regular representation of G gives an injection $G \to S_{|G|}$. This is an isomorphism onto its image. This is Cayley's Theorem.

Def: Let $H \leq G$. A subset $\{g_1, \ldots, g_k\} \subseteq G$ is a transversal of H if $\mathcal{H} = \{g_1 H, \ldots, g_k H\}$ is a complete set of distinct cosets of H in G. In that case, $k = [G:H]$, the index of H in G.

Def: Given $H \leq G$ and a transversal $\mathcal{H}$ of H, the coset representation of G with respect to H is $\mathbb{C}\mathcal{H} = \{c_1 \overrightarrow{g_1 H} + \ldots + c_n \overrightarrow{g_n H} \mid c_i \in \mathbb{C}\}$. It is a vector space under vector addition and scalar multiplication with basis $\{\overrightarrow{g_1 H}, \ldots, \overrightarrow{g_n H}\}$. It is a G-module under the action

$$g\left(\sum c_i \overrightarrow{g_i H}\right) = \sum c_i \overrightarrow{(g g_i) H}$$

Ex: $G = S_3$, $H = \langle (2\,3) \rangle = \{(1), (2\,3)\}$, $\mathcal{H} = \{H, (1\,2)H, (1\,3)H\}$

$\mathbb{C}\mathcal{H} = \{c_1 \vec{H} + c_2 \overrightarrow{(1\,2)H} + c_3 \overrightarrow{(1\,3)H}\}$, $\quad (1\,3)\vec{H} = \overrightarrow{(1\,3)H}, (1\,3)\overrightarrow{(1\,2)H} = \overrightarrow{(1\,2)H}, (1\,3)\overrightarrow{(1\,3)H} = \vec{H}$

Thus, $\quad X(1\,3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Def: Let V be a G-module. A <u>submodule</u> of V is a subspace W of V which is closed under the action of G, i.e. $\forall g \in G$, $\vec{w} \in W$, $g\vec{w} \in W$. We say W is a <u>G-invariant</u> subspace.

Remark: A submodule of V is a subset of V which is a G-module under the same operations

Note: G acts on V denoted $G \circlearrowright V$.

Def: Let V be a G-module. The <u>trivial submodules</u> are $\vec{W} = \{\vec{0}\}$ and $W = V$. $W = V$ is not a <u>proper submodule</u>.

Ex: Defining representation on S_n. $V = \mathbb{C}[\vec{1}, \vec{2}, \ldots, \vec{n}] = \{c_1\vec{1} + \ldots + c_n\vec{n} \mid c_i \in \mathbb{C}\}$
Define W by $W = \langle \vec{1} + \vec{2} + \ldots + \vec{n} \rangle$. Then W is a nontrivial S_n submodule. It is a subspace and is fixed by S_n.

Notation: $W \leq V$

Q: What representation is W? What is $X(\pi)$? $X(\pi) = (1)$, the trivial representation.

Def: If G fixes every element of W, we say <u>G acts trivially on W</u>, i.e.

$$g\vec{w} = \vec{w}, \ \forall g \in G, \ \vec{w} \in W$$

Def: A nonzero G-module V is <u>reducible</u> if it contains a nontrivial submodule. Otherwise, V is <u>irreducible</u>.

Prop: A nonzero G-module is reducible iff it has a basis B such that, $\forall g \in G$, $\exists$ matrices $A(g)$, $B(g)$, $C(g)$ where $A(g)$ is square of size independent of g with

$$X(g) = \left(\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array}\right)$$

Pf: $(\Rightarrow)$ Let V be a reducible nonzero G-module of $\dim(V) = d$. Let W be a nontrivial submodule of $\dim(W) = f$, $0 < f < d$. Let W have basis $\{\vec{w}_1, \ldots, \vec{w}_f\}$. Define $B = \{\vec{w}_1, \ldots, \vec{w}_f, \vec{v}_{f+1}, \ldots, \vec{v}_d\}$ to be a basis of V with $\vec{v}_i \in V$. Then

$$X(g) = \left([g\vec{w}_1]_B \, [g\vec{w}_2]_B \cdots [g\vec{v}_d]_B\right)$$

But $g\vec{w}_i \in W$ since $W \leq V$. Thus, its coordinates in positions $f+1, \ldots, d$ are 0. There exists a nonempty block matrix of 0's. $A(g)$, $B(g)$, and $C(g)$ are found by restricting G to W. $A(g)$ is an $f \times f$ matrix.

$(\Leftarrow)$ Suppose

$$X(g) = \left(\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array}\right)$$

Let $V = \mathbb{C}^d$, i.e. $X: G \to GL_d$. $W = \mathbb{C}\{\vec{e}_1, \ldots, \vec{e}_f\}$.

Ex: The defining representation of S_n is reducible. $W = \langle \vec{1} + \dots + \vec{n} \rangle$.

Q: What is the corresponding matrix representation of $G = S_3$? Can we show that $V = \mathbb{C}\{1,2,3\}$ is not irreducible by constructing a basis B?

We begin with the basis of $W = \{\vec{1} + \vec{2} + \vec{3}\}$ and extend to a basis of V. Let $B = \{\vec{1} + \vec{2} + \vec{3}, \vec{2}, \vec{3}\}$. Then B is a basis of V. We claim $X(g)$ will have the desired form $\forall g \in G$. Let $\sigma = (1\,3) \in S_3$. We compute $X(1\,3)$.

$$(1\,3)(\vec{1} + \vec{2} + \vec{3}) = \vec{3} + \vec{2} + \vec{1} = \vec{1} + \vec{2} + \vec{3}$$
$$(1\,3)(\vec{2}) = \vec{2}$$
$$(1\,3)(\vec{3}) = \vec{1} = (\vec{1} + \vec{2} + \vec{3}) - \vec{2} - \vec{3}$$
$$\Rightarrow X(1\,3) = \left(\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & 1 & -1 \\ 0 & 0 & -1 \end{array}\right)$$

The other permutations will have a similar form.

Def: Let U, W be subspaces of a vector space V. Then V is the <u>(internal) direct sum</u> of U and W if $\forall \vec{v} \in V$, $\exists! (\vec{u}, \vec{w}) \in U \times W$ such that $\vec{v} = \vec{u} + \vec{w}$. We write $V = U \oplus W$

Def: Let V be a G-module and $V = U \oplus W$. If U and W are submodules then we say U and W are <u>complements of each other</u>.

Ex: Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ be the group of nonzero complex numbers under multiplication. Then define $\mathbb{C}^\times \circlearrowright \mathbb{C}^2$ by $\lambda(x,y) = (\lambda x, \lambda y)$, $\lambda \in \mathbb{C}^\times$, $(x,y) \in \mathbb{C}^2$. Then $\mathbb{C}^2$ is a $\mathbb{C}^2$-module.

1) $1(x,y) = (1x, 1y) = (x,y)$
2) $\lambda(\mu(x,y)) = (\lambda\mu)(x,y) = (\lambda\mu x, \lambda\mu y)$
3) $\lambda(c\vec{v} + d\vec{w}) = c\lambda\vec{v} + d\lambda\vec{w}$
4) $\lambda\vec{v} \in \mathbb{C}^2$, $\forall \lambda \in \mathbb{C}^\times$, $\vec{v} \in \mathbb{C}^2$

Let $U = \{(x,0) \in \mathbb{C}^2\}$ and $W = \{(0,y) \in \mathbb{C}^2\}$. Note $\mathbb{C}^2 = U \oplus W$. Also note $\forall (x,0) \in U$, $(0,y) \in W$, $\lambda \in \mathbb{C}^\times$, $\lambda(x,0) = (\lambda x, 0) \in U$ and $\lambda(0,y) = (0, \lambda y) \in W$. Thus, U and W are submodules and complements of each other.

Def: The matrix X is the <u>direct sum</u> of matrices A and B if $X = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$ as a block matrix. We write $X = A \oplus B$.

Ex: $V = \mathbb{C}\{1,2,3\}$, $U = \langle \vec{1} + \vec{2} + \vec{3} \rangle$, $W = \langle \vec{2}, \vec{3} \rangle$, $V = U \oplus W$ as vector spaces but $(1\,3)(\vec{3}) = \vec{1} \notin W$ so W is not a submodule.

Q: How can we find the complement of U? We use inner products.

Def: An inner product on a vector space V over $\mathbb{C}$ is a positive-definite Hermitian form, i.e. it is a map $\langle \cdot,\cdot \rangle : V \times V \to \mathbb{C}$ such that

1) $0 \leq \langle \vec{v},\vec{v}\rangle \in \mathbb{R}, \forall \vec{v} \in V$ and $\langle \vec{v},\vec{v}\rangle = 0 \iff \vec{v} = 0$ (positive definite)
2) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ where $\bar{z}$ is the $\mathbb{C}$-conjugate of z (Hermitian symmetry)
3) $\langle cx,y\rangle = c\langle x,y\rangle$, $\langle x+z,y\rangle = \langle x,y\rangle + \langle z,y\rangle$ (linear in the 1^{st} factor)

Note: $\langle x, cy\rangle = \overline{\langle cy,x\rangle} = \bar{c}\overline{\langle y,x\rangle} = \bar{c}\langle x,y\rangle$ (sesquilinear)

$\langle x,y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

Ex: Let V be a vector space basis $B = \{\vec{u}_1,\dots,\vec{u}_n\}$ with

$$\vec{v} = a_1\vec{u}_1 + a_2\vec{u}_2 + \dots + a_n\vec{u}_n \in V, \quad \vec{w} = b_1\vec{u}_1 + b_2\vec{u}_2 + \dots + b_n\vec{u}_n \in V$$

Define

$$\langle \vec{v},\vec{w}\rangle = a_1\bar{b}_1 + a_2\bar{b}_2 + \dots + a_n\bar{b}_n$$

Def: A vector space is an <u>inner product space</u> if it contains an inner product.

Thm: (Maschke's Theorem) Let G be a finite group and V be a nonzero G-module. Then $\exists$ irreducible representations $W^{(1)},\dots,W^{(k)}$ such that

$$V = W^{(1)} \oplus W^{(2)} \oplus \dots \oplus W^{(k)}$$

Proof: We induct on $d = \dim(V)$. When $d=1$, V is irreducible with $k=1$, $W^{(1)} = V$. Now, let $d>1$ and assume the result holds for all G-modules of dimension $<d$. If V is irreducible then we are done. Otherwise, it contains a nontrivial submodule W where $0<\dim(W)<d$. We will split $V = W \oplus W^{\perp}$. Define $\langle \vec{v}_i,\vec{v}_j\rangle = \delta_{i,j}$ where $\{\vec{v}_1,\dots,\vec{v}_d\}$ is a basis of V. Now define

$$\langle \vec{v},\vec{w}\rangle' = \sum_{g\in G}\langle g\vec{v},g\vec{w}\rangle$$

Note for $h\in G$, $\langle h\vec{v},h\vec{w}\rangle' = \sum_{g\in G}\langle gh\vec{v},gh\vec{w}\rangle = \sum_{g'\in G}\langle g'\vec{v},g'\vec{w}\rangle = \langle \vec{v},\vec{w}\rangle'$ so it respects the action of G. Define

$$W^{\perp} = \{\vec{v}\in V \mid \langle \vec{v},\vec{w}\rangle' = 0, \forall \vec{w}\in W\}$$

Then this is a nontrivial submodule complementary to W, i.e. $V = W\oplus W^{\perp}$. But $0<\dim(W),\dim(W^{\perp})<d$. Thus, W and $W^{\perp}$ can be written as a direct sum of irreducible submodules. Thus, Maschke's Theorem holds.

Cor: Let G be finite and let $X: G \to GL_d$ be a matrix representation of dimension $d>0$. Then $\exists$ a fixed matrix T such that $\forall g\in G$, $X(g)$ has the form

$$TX(g)T^{-1} = \begin{pmatrix} X^{(1)}(g) & & 0 \\ & X^{(2)}(g) & \\ 0 & & \ddots \; X^{(k)}(g) \end{pmatrix}$$

where $X^{(i)}(g)$ is a block and is an irreducible matrix representation of G for all $i = 1, \dots, k$.

Pf: Let $V = \mathbb{C}^d$ with G-module structure given by $(g, \vec{v}) = g\vec{v} = X(g)\vec{v}$. By Maschke's Theorem, $\exists$ irreducible submodules $W^{(1)}, \dots, W^{(k)}$ such that $V = W^{(1)} \oplus \dots \oplus W^{(k)}$ and $\dim(W^{(i)}) = d_i$. Then V admits a basis of size $d = \sum_{i=1}^{k} d_i$ where the first d_1 elements form a basis of $W^{(1)}$, etc. Define $T : \mathbb{C}^d \to \mathbb{C}^d$ by $T(e_i) = b_i$ where e_i is the i^{th} standard basis vector of $\mathbb{C}^d$ and b_i is the i^{th} basis vector of V. Thus T is an isomorphism and is invertible. Then $TX(g)T^{-1}$ expresses the action of $X(g)$ in the basis of V and each $W^{(i)}$ is closed under G.

Def: A representation is <u>completely reducible</u> if it can be written as a direct sum of irreducible representations.

Def: Let V, W be G-modules. A linear transformation $\varphi : V \to W$ is a <u>G-homomorphism</u> or a homomorphism of G-modules if $\forall g \in G$, $\vec{v} \in V$,

$$\varphi(g\vec{v}) = g\varphi(\vec{v})$$

Remark: Let $B = \{\vec{v}_1, \dots, \vec{v}_d\}$ be a basis for V and $C = \{\vec{w}_1, \dots, \vec{w}_f\}$ a basis for W. Let $T = [\varphi]_{B \to C}$. Then T is $f \times d$. Let X, Y be matrix representations of V and W, respectively. Since φ is a G-homomorphism, $\forall g \in G$, $\vec{v} \in V$,

$$TX(g)(\vec{v}) = Y(g)T(\vec{v})$$

Since this holds $\forall \vec{v} \in V$, we have $TX(g) = Y(g)T$. Hence

$$\begin{tikzcd} \mathbb{C}^d \arrow[r, "X(g)"] \arrow[d, "T"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{C}^d \arrow[d, "T"] \\ \mathbb{C}^f \arrow[r, "Y(g)"'] & \mathbb{C}^f \end{tikzcd}$$

Ex: $W = \mathbb{C}\{\vec{1}, \dots, \vec{n}\}$ defining representation of S_n and not irreducible. $\langle \vec{1} + \dots + \vec{n} \rangle$ fixed by S_n. Define $V = \mathbb{C}\{\vec{v}\}$ ($\cong \mathbb{C}$) and $\varphi : V \to W$ by $\vec{v} \mapsto \vec{1} + \dots + \vec{n}$. This yields a linear transformation. Note $\forall \sigma \in S_n$

$$\varphi(\sigma(\vec{v})) = \varphi(\vec{v}) = \vec{1} + \dots + \vec{n}$$
$$\sigma\varphi(\vec{v}) = \sigma(\vec{1} + \dots + \vec{n}) = \vec{1} + \dots + \vec{n}$$

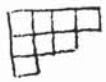
Thus, φ is an S_n-homomorphism.

Def: Let V, W be G-modules. A <u>G-isomorphism</u> or isomorphism of G-modules or an equivalence of G-modules is a bijective G-homomorphism. We say V and W are G-isomorphic or equivalent. We write $V \cong W$.

Ex: Let $G=\{g, g^2=e\}=C_2=\mathbb{Z}/2\mathbb{Z}$. Let $V=\mathbb{C}[G]=\{c_1 g+c_2 e\}$ and $W=\mathbb{C}^2=\langle e_1,e_2\rangle$. We give W the structure of a G-module by $\vec e\,\vec e_i=\vec e_i$, $g\vec e_1=\vec e_2$, $g\vec e_2=\vec e_1$. We claim $V\cong W$. Define $\Theta: V\to W$ by $\Theta(\vec e)=\vec e_1$ and $\Theta(\vec g)=\vec e_2$. We check

$$\Theta(g\vec g)=\Theta(\vec e)=\vec e_1$$
$$g\Theta(\vec g)=g\vec e_2=\vec e_1$$

Def: If $\lambda\vdash n$ is a partition of n, $\lambda=(\lambda_1,\dots,\lambda_\ell)$, $\lambda_1\geq\dots\geq\lambda_\ell$. Then a <u>Young diagram</u> of shape λ has λ_i boxes in row i.

Ex: $\lambda=(4,3,1)$ 

Def: A <u>Young tableau</u> of shape λ is a Young diagram of shape λ with entries in each box from the set $\{1,\dots,n\}$.

Def: Two Young tableaux are <u>row equivalent</u> if they have the same entries in each row, possibly in different order.

Ex: $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}\sim\begin{array}{|c|c|c|}\hline 3&4&1\\\hline 2\\\cline{1-1}\end{array}$

Def: A <u>Young tabloid</u> is a row equivalence class of Young tableaux.

Ex: $\begin{array}{c}\hline 1\ 3\ 4\\\hline 2\\\hline\end{array}=\begin{array}{c}\hline 4\ 3\ 1\\\hline 2\\\hline\end{array}$

Ex: S_3, $\mathbb{C}\{\vec 1,\vec 2,\vec 3\}$ and $H=\langle(2\,3)\rangle$ with $\mathcal{H}=\{H,(12)H,(13)H\}$ and coset representation of S_3 with respect to $\mathcal{H}$, $\mathbb{C}\mathcal{H}=\{c_1\vec H+c_2\overrightarrow{(12)H}+c_3\overrightarrow{(13)H}\}$. Let S denote the set of Young tabloids of shape $(2,1)$.

$$S=\left\{\begin{array}{c}\hline 1\ 2\\\hline 3\\\hline\end{array},\ \begin{array}{c}\hline 1\ 3\\\hline 2\\\hline\end{array},\ \begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array}\right\}$$

S admits an S_3 action where for $\sigma\in S_3$, $\sigma\left(\begin{array}{c}\hline i\ j\\\hline k\\\hline\end{array}\right)=\begin{array}{c}\hline \sigma(i)\ \sigma(j)\\\hline \sigma(k)\\\hline\end{array}$. We claim $\mathbb{C}S\cong\mathbb{C}\mathcal{H}$. Define $\mathbb{C}\mathcal{H}\xrightarrow{\Theta}\mathbb{C}S$ by

$$\Theta(\vec H)=\begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array}\qquad\Theta(\overrightarrow{(12)H})=(12)\begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array}=\begin{array}{c}\hline 1\ 3\\\hline 2\\\hline\end{array}\qquad\Theta(\overrightarrow{(13)H})=(13)\begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array}=\begin{array}{c}\hline 1\ 2\\\hline 3\\\hline\end{array}$$

This maps basis to basis and hence is a vector space isomorphism. Also, Θ is S_3-invariant since

$$\Theta((13)\vec H)=\Theta(\overrightarrow{(13)H})=\begin{array}{c}\hline 1\ 2\\\hline 3\\\hline\end{array}=(13)\begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array}=(13)\,\Theta(\vec H)$$

Thus, $\mathbb{C}S\cong\mathbb{C}\mathcal{H}$. We also claim $\mathbb{C}S\cong\mathbb{C}\{1,2,3\}$. Define $\varphi:\mathbb{C}\{\vec 1,\vec 2,\vec 3\}\to\mathbb{C}S$ by

$$\varphi(\vec 1)=\begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array}\qquad\varphi(\vec 2)=\begin{array}{c}\hline 1\ 3\\\hline 2\\\hline\end{array}\qquad\varphi(\vec 3)=\begin{array}{c}\hline 1\ 2\\\hline 3\\\hline\end{array}$$

Then φ is S_3-invariant: $(1\,2\,3)\varphi(\vec 1)=(1\,2\,3)\begin{array}{c}\hline 2\ 3\\\hline 1\\\hline\end{array}=\begin{array}{c}\hline 3\ 1\\\hline 2\\\hline\end{array}=\varphi(\vec 2)=\varphi((1\,2\,3)\vec 1)$

Therefore $\mathbb{C}\{\vec 1,\vec 2,\vec 3\}\cong\mathbb{C}\mathcal{H}$.

Thm: (Schur's lemma) Let V and W be G-modules. If V and W are irreducible and $\theta : V \to W$ is a G-homomorphism, then θ is either the zero map or a G-isomorphism.

Pf: Note $\ker\theta \le V$. But V is irreducible so $\ker\theta$ is trivial and either $\ker\theta = \{\vec{0}\}$ or $\ker\theta = V$. Also, $\operatorname{im}\theta = \{0\}$ or $\operatorname{im}\theta = W$. If $\ker\theta = V$ or $\operatorname{im}\theta = \{\vec{0}\}$ then θ is the zero map. Otherwise, $\ker\theta = \{\vec{0}\}$ and $\operatorname{im}\theta = W$. Then θ is bijective and hence an isomorphism.

Remark: This proof works for arbitrary fields and infinite groups.

Cor: Let X and Y be irreducible matrix representations. If T is a matrix s.t. $TX = YT$, then T is invertible or the zero matrix.

Cor: Let X be irreducible. Suppose $TX(g) = X(g)T$ for all $g \in G$. Then $T = cI$, $c \in \mathbb{C}$.

Pf: If $TX = XT$, then $TX - cX = XT - cX$ and $(T - cI)X = X(T - cI)$. Thus, $T - cI$ is the zero matrix or is invertible. But taking c to be an eigenvalue shows $T - cI$ is not invertible. Hence $T - cI = 0$.

Cor: Let V, W be G-modules. Let V be irreducible. Then $\dim \operatorname{Hom}_G(V, W) = 0$ if and only if W contains no submodules isomorphic to V.

Def: Let R be a commutative ring with identity. A (left) <u>R-module</u> is an abelian group M together with a (left) action (the R-module structure on M) $R \times M \to M$, $(r, x) \mapsto rx$ such that

(1) $r(sx) = (rs)x$ (mixed associativity)
(2) $r(x+y) = rx + ry$, $(r+s)x = rx + sx$ (distributivity)
(3) $1 \cdot x = x$ (unital)

for all $x, y \in M$, $r, s \in R$.

Ex: If K is a field, a unital K-module is a vector space, and conversely.

Note: If A is an abelian group, its ring of endomorphisms is denoted $\operatorname{End}_{\mathbb{Z}}(A)$ where for $\varphi, \psi : A \to A$, $(\varphi + \psi)(a) = \varphi(a) + \psi(a)$ and $(\varphi \circ \psi)(a) = \varphi(\psi(a))$.

Note: There is a one-to-one correspondence between left R-modules over M (fixed) and ring homomorphisms.

left R-modules		ring homomorphisms			
$R \times M \to M$	$\longleftrightarrow$	$R \to \operatorname{End}_{\mathbb{Z}}(M)$	$(r, x) \mapsto \varphi_r(x)$	$\longleftrightarrow$	$r \mapsto \varphi_r : M \to M$, $x \mapsto rx$

<u>Def:</u> Let M, N be R-modules. An <u>R-homomorphism</u> or <u>hom of R-modules</u> is a hom of abelian groups which respects the action of R, i.e. it is a map $\varphi: M \to N$ such that

$$\varphi(x+y) = \varphi(x) + \varphi(y) \;,\; \varphi(rx) = r\varphi(x) \;,\; \forall r \in R,\ x,y \in M$$

<u>Def:</u> Let A, B, C be R-modules. A <u>bilinear map</u> $\beta: A \times B \to C$ is a map such that

(1) $\beta(a+a', b) = \beta(a,b) + \beta(a',b)$

(2) $\beta(a, b+b') = \beta(a,b) + \beta(a,b')$

(3) $\beta(ra, b) = r\beta(a,b) = \beta(a, rb)$

$\forall a, a' \in A,\ b, b' \in B,\ r \in R$

<u>Note:</u> The set of all R-module homomorphisms $M \to N$ is denoted $\mathrm{Hom}_R(M,N)$

<u>Lemma:</u> Let A, B, C be R-modules. Let $\beta: A \times B \to C$ be a map of sets. The following are equivalent:

(1) β is bilinear

(2) $a \mapsto \beta(a, -)$ is an R-homomorphism $A \to \mathrm{Hom}_R(B, C)$

(3) $b \mapsto \beta(-, b)$ is an R-homomorphism $B \to \mathrm{Hom}_R(A, C)$

<u>Def:</u> Let A and B be R-modules. The <u>tensor product</u> of A and B is an R-module $A \otimes_R B$ together with a bilinear map

$$\tau: A \times B \to A \otimes_R B$$

such that the following universal property holds. For every R-module C and bilinear map $\beta: A \times B \to C$ there exists a unique R-homomorphism $A \otimes_R B \to C$ such that

$$\begin{tikzcd} A \times B \arrow[d, "\tau"'] \arrow[dr, "\beta"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"] & \\ A \otimes_R B \arrow[r, dotted] & C \end{tikzcd}$$

this diagram commutes.

<u>Prop:</u> For any R-modules A and B, $A \otimes_R B$ exists and is unique up to isomorphism.

<u>Proof:</u> Suppose $(A \otimes_R B)'$, $\tau': A \times B \to (A \otimes_R B)'$ is another tensor product. Then

$$\begin{tikzcd} A \times B \arrow[d, "\tau"'] \arrow[dr, "\tau'"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"] & \\ A \otimes_R B \arrow[r, "\varphi"'] & (A \otimes_R B)' \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[d, "\tau'"'] \arrow[dr, "\tau"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"] & \\ (A \otimes_R B)' \arrow[r, "\psi"'] & A \otimes_R B \end{tikzcd}$$

$$\tau' = \varphi \circ \tau = \varphi \circ \psi \circ \tau' \qquad \tau = \psi \circ \tau' = \psi \circ \varphi \circ \tau$$

Then $\psi = \varphi^{-1}$ and hence

$$A \otimes_R B \cong (A \otimes_R B)'$$

Thus, if $A \otimes_R B$ exists, it is unique up to isomorphism. We now construct it.

Let T be the R-module generated by all pairs $(a,b) \in A \times B$ subject to the relations

$$(a+a', b) = (a,b) + (a',b) \quad (a, b+b') = (a,b) + (a,b') \quad (ra, b) = (a, rb)$$

Then the map $A \times B \to T$ is bilinear $(a,b) \mapsto [(a,b)]$.

Note: τ is called the <u>tensor map</u>, $\tau(a,b) = a\otimes b$

Remark: $T \cong A\otimes_R B$ is generated by elements of the form $a\otimes b$ and

$$(a+a')\otimes b = a\otimes b + a'\otimes b \qquad a\otimes(b+b') = a\otimes b + a\otimes b'$$

$$(ra)\otimes b = r(a\otimes b) = a\otimes(rb)$$

$A\otimes_R B = \langle a\otimes b \mid a\in A, b\in B\rangle$ or sums of $a\otimes b$ and inverses.

$A\otimes_R B$ contains proper sums, not all elements are of the form $a\otimes b$.

Ex: V $\mathbb{C}$-vector space, basis $=\{\vec{v}_1,\dots,\vec{v}_d\}$, W with basis $\{\vec{w}_1,\dots,\vec{w}_f\}$

$V\otimes_{\mathbb{C}} W = \langle \vec{v}\otimes\vec{w} \mid \vec{v}\in V, \vec{w}\in W\rangle = \langle \vec{v}_i\otimes\vec{w}_j \mid 1\le i\le d, 1\le j\le f\rangle$

i.e. $\{\vec{v}_i\otimes\vec{w}_j\}$ is a basis for $V\otimes_{\mathbb{C}} W$ so $\dim(V\otimes_{\mathbb{C}} W) = d\times f$, $\dim(V\oplus W) = d+f$

Ex: For matrices X, Y their tensor product is

$$X\otimes Y = (x_{ij}Y) = \begin{pmatrix} x_{11}Y & x_{21}Y & \cdots \\ x_{12}Y & \ddots & \\ \vdots & & \end{pmatrix}$$

Lemma: $A, X\in \mathrm{Mat}_d$, $B, Y\in\mathrm{Mat}_f$

(1) $(A\oplus B)(X\oplus Y) = AX\oplus BY$

(2) $(A\otimes B)(X\otimes Y) = AX\otimes BY$

Def: The <u>commutant algebra</u> or <u>commutator algebra</u> of a matrix representation $X: G\to GL_d$ is

$$\mathrm{Com}\, X = \{T\in\mathrm{Mat}_d \mid TX(g) = X(g)T, \ \forall g\in G\}$$

Def: The <u>endomorphism algebra</u> or <u>commutator algebra</u> of the G-module V is

$$\mathrm{End}\, V = \{\theta : V\to V \mid \theta \text{ is a } G\text{-homomorphism}\}$$

Thm: Let V be a G-module such that $V \cong m_1V^{(1)}\oplus m_2V^{(2)}\oplus\dots\oplus m_kV^{(k)}$ where $V^{(i)}$ are inequivalent and irreducible and m_i is the multiplicity of $V^{(i)}$, i.e. $m_iV^{(i)} = V^{(i)}\oplus V^{(i)}\oplus\dots\oplus V^{(i)}$ (m_i times). Then,

1) $\dim V = \sum_{i=1}^{k} m_i d_i$ where $\dim(V^{(i)}) = d_i$

2) $\mathrm{End}(V) \cong \bigoplus_{i=1}^{k} \mathrm{Mat}_{m_i}$

3) $\dim(\mathrm{End}(V)) = \sum_{i=1}^{k} m_i^2$

4) The center $Z_{\mathrm{End}(V)}$ is isomorphic to the algebra of diagonal matrices of degree k.

5) $\dim Z_{\mathrm{End}(V)} = k$

Pf: (1) Each copy of $V^{(i)}$ has d_i basis elements. The union is a basis of V.
(2) This follows from Schur's lemma.
(3) $\mathrm{Mat}_{m_i} \cong \mathbb{C}^{m_i^2}$
(4) Only diagonal matrices commute.
(5) $\{k \times k$ diagonal matrices$\} \cong \mathbb{C}^k$

Def: Let $X: G \to GL_d$ be a matrix representation. The character of X is

$$\chi: G \to \mathbb{C} \text{ where } \chi(g) = \mathrm{Tr}(X(g))$$

If V is a G-module, the character of V is the character of any matrix representation of V.

Rmk: If $A = (a_{ij}) \in \mathrm{Mat}_n$ then $\mathrm{Tr}(A) = \sum_{i=1}^n a_{ii} = a_{11} + \dots + a_{nn}$. The character of V is independent of basis because $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ so $\mathrm{Tr}(TXT^{-1}) = \mathrm{Tr}(T^{-1}TX) = \mathrm{Tr}(X)$. Also, $\mathrm{Tr}(X)$ = sum of eigenvalues counted with multiplicity.

Ex: Defining representation S_3: $\chi(12) = 1$, $\chi(1) = 3$, $\chi(123) = 0$

Claim: For $\pi \in S_n$, $\chi(\pi) = \#$ fixed points of π

Pf: $\chi(\pi) = \#$ 1's on the diagonal, $(X(\pi))_{ii} = 1 \iff \pi(i) = i$

Ex: Compute χ^{reg} for an arbitrary finite group. $G = \{g_1, \dots, g_n\}$, $\mathbb{C}[G] = V$ in regular representation. Recall a matrix representation of $\mathbb{C}[G]$ consists of permutation matrices. We compute $X(e) = \mathrm{Tr}\, I_n = n = |G|$. Now let $g \neq e$. $\chi(g) = \mathrm{Tr}(X(g))$. Since $g \cdot g_i = g_i \Rightarrow g = e$, $g \cdot g_i \neq g_i$ for all i and $\chi(g) = 0$. Thus,

$$\chi(g) = \begin{cases} 0, & g \neq e \\ |G|, & g = e \end{cases}$$

Prop: Let $X: G \to GL_d$.
(1) $\chi(e) = d$
(2) $g = khk^{-1} \Rightarrow \chi(g) = \chi(h)$
(3) Let X, Y have characters χ and Ψ, respectively. Then $X \cong Y \Rightarrow \chi(g) = \Psi(g)$, $\forall g \in G$

Def: A class function on a group G is a map $f: G \to \mathbb{C}$ such that $f(g) = f(h)$ if g and h are in the same conjugacy class. The set of class functions on G is denoted $R(G)$.

Claim: $R(G)$ is an inner product space.

Pf: $f_1, f_2 \in R(G)$. $(f_1 + f_2)(g) = f_1(g) + f_2(g)$. For $c \in \mathbb{C}$, $(cf)(g) = c(f(g))$. Then $R(G)$ is a vector space over $\mathbb{C}$. Define for a conjugacy class K in G

$$f_k(g) = \begin{cases} 1, & g \in K \\ 0, & g \notin K \end{cases}$$

Then $\{f_k\}$ forms a basis of $R(G)$. Thus, $\dim R(G) = \#$ distinct conjugacy classes in G. Let $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$. It satisfies the axioms for an inner product.

Notation: $\chi_K = \chi(g)$, $g \in K$

<u>Def:</u> The <u>character table</u> of G is

rows correspond to irreducible representations

	$\cdots K \cdots$
$\vdots$	$\vdots$
χ	$\cdots \; \chi_K$
$\vdots$	

columns indexed by conjugacy classes

<u>Note:</u> $X \cong Y \iff \chi = \psi$ (the converse is true!)

<u>Ex:</u> The cyclic group of order n, C_n. There are n conjugacy classes because C_n is abelian. Thus, $\exists n$ inequivalent irreducible representations of C_n. Each of the n^{th} roots of unity gives a linear hence irreducible representation. These are thus all the irreducible representations of C_n. Consider $n=4$.

C_4	e	g	g^2	g^3
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	i	-1	$-i$	1
$\chi^{(3)}$	-1	1	-1	1
$\chi^{(4)}$	$-i$	-1	i	1

<u>Ex:</u> S_3: conjugacy classes in S_3 correspond to .

S_3	K_1	K_2	K_3
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	-1	1
$\chi^{(3)}$	?	?	?

<u>Note:</u> We will use inner products to find the missing irreducible representation.

<u>Remark:</u> The function space $N_F = \{f : G \to \mathbb{C}\}$ is an inner product space under pointwise addition, scalar multiplication, and the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

<u>Prop:</u> Let χ, ψ be characters of G. Then

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \psi(g^{-1})$$

<u>Pf:</u> Let ψ be the character of V. Then V admits an orthonormal basis. Then the matrix representation Y of V has these columns. Hence Y is unitary, i.e. $Y^* Y = I$, i.e. $Y^{-1} = \overline{Y}^T$. Then $\overline{\psi(g)} = \mathrm{Tr}(\overline{Y(g)}) = \mathrm{Tr}((Y(g)^{-1})^T) = \mathrm{Tr}(Y(g^{-1})^T) = \mathrm{Tr}(Y(g^{-1})) = \psi(g^{-1})$.

<u>Thm:</u> (Character Relations of the First Kind) Let χ, ψ be characters of G, irreducible. Then,

$$\langle \chi, \psi \rangle = \delta_{\chi,\psi} = \begin{cases} 1, & \chi = \psi \\ 0, & \chi \neq \psi \end{cases}$$

<u>Pf:</u> Let $\chi = \mathrm{Tr}\, A$, $\psi = \mathrm{Tr}\, B$, $A \in \mathrm{Mat}_d$, $B \in \mathrm{Mat}_f$. Let $X = (x_{ij}) \in \mathrm{Mat}_{d \times f}$ be a matrix of variables. Define

$$Y = \frac{1}{|G|} \sum_{g \in G} A(g) X B(g^{-1})$$

We claim $A(h) Y = Y B(h)$, $\forall h \in G$. Note that

$$\begin{aligned} A(h) Y B(h^{-1}) &= \frac{1}{|G|} \sum_{g \in G} A(h) A(g) X B(g^{-1}) B(h^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} A(hg) X B((hg)^{-1}) \\ &= \frac{1}{|G|} \sum_{\substack{\tilde{g} \in G \\ \tilde{g} = hg}} A(\tilde{g}) X B(\tilde{g}) \\ &= Y \end{aligned}$$

Thus, by Schur's lemma, $Y = \begin{cases} 0, & A \not\cong B \\ cI_d, & A \cong B \end{cases}$

<u>Case $\chi \neq \psi$</u>: Then $A \not\cong B$ and $Y = 0$, i.e. $y_{ij} = 0 \ \forall i, j$.

$$y_{ij} = \frac{1}{|G|} \sum_{g \in G} \sum_{k,\ell} a_{ik}(g) x_{k\ell} b_{\ell j}(g^{-1}) = 0$$

Then this is the zero polynomial so each coefficient is 0 and

$$0 = \frac{1}{|G|} \sum_{g \in G} a_{ik}(g) b_{\ell j}(g^{-1}) \quad \forall i, j, k, \ell$$

i.e. $\langle a_{ik}, b_{\ell j} \rangle = 0$. Let $k = i$ and $\ell = j$. Then $\langle a_{ii}, b_{jj} \rangle = 0, \forall i, j$. Thus,

$$0 = \sum_{ij} \langle a_{ii}, b_{jj} \rangle = \langle \chi, \psi \rangle$$

<u>Case $\chi = \psi$</u>: wlog $A = B$. Then $Y = cI_d$ for some $c \in \mathbb{C}$. Then $y_{ij} = c\delta_{ij}$. For $i \neq j$, $\langle a_{ik}, b_{\ell j} \rangle = \langle a_{ik}, a_{\ell j} \rangle = 0$. For $i = j$, $Y = cI_d = \frac{1}{|G|} \sum_{g \in G} A(g) X B(g^{-1})$. Thus,

$$\begin{aligned} \operatorname{Tr} Y = cd &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(A(g) X B(g^{-1})) \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(X B(g^{-1}) A(g)) \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(X A(g^{-1}) A(g)) \\ &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr} X \\ &= \operatorname{Tr} X \end{aligned}$$

$$y_{ii} = c = \frac{1}{d} \operatorname{Tr} X = \frac{1}{|G|} \sum_{k,\ell} \sum_{g \in G} a_{ik}(g) x_{k\ell} a_{\ell i}(g^{-1}) = \frac{1}{d}(x_{11} + \dots + x_{dd})$$

Thus, $\langle a_{ik}, a_{\ell i} \rangle = \frac{1}{|G|} \sum_{g \in G} a_{ik}(g) a_{\ell i}(g^{-1})$
$= \frac{1}{d} \delta_{k,\ell}$

Thus, $\langle \chi, \chi \rangle = \sum_{i,j} \langle a_{ii}, a_{jj} \rangle = \sum_i \langle a_{ii}, a_{ii} \rangle = \sum_{i=1}^{d} \frac{1}{d} = 1$.

<u>Cor</u>: Let X be a matrix representation of G with character χ. Suppose $X \cong m_1 X^{(1)} \oplus \dots \oplus m_k X^{(k)}$ where $X^{(i)}$ are inequivalent irreducible representations of G. Then

1) $\chi = m_1 \chi^{(1)} + \dots + m_k \chi^{(k)}$
2) $\langle \chi, \chi^{(i)} \rangle = m_i, \ \forall i = 1, \dots, k$
3) $\langle \chi, \chi \rangle = m_1^2 + \dots + m_k^2$
4) X is irreducible $\iff \langle \chi, \chi \rangle = 1$
5) Let Y be another representation of G with character ψ. Then $X \cong Y \iff \chi = \psi \quad (\chi(g) = \psi(g) \ \forall g \in G)$

Pf: 1) $\chi = \operatorname{Tr} X = \operatorname{Tr}\left(\bigoplus_{i=1}^{k} m_i X^{(i)}\right) = \sum_{i=1}^{k} \operatorname{Tr}(m_i X^{(i)}) = \sum_{i=1}^{k} m_i \chi^{(i)}$

2) $\langle \chi, \chi^{(i)} \rangle = \langle \sum m_i \chi^{(i)}, \chi^{(i)} \rangle = \sum_{j=1}^{k} m_j \langle \chi^{(j)}, \chi^{(i)} \rangle = m_i$

3) $\langle \chi, \chi \rangle = \langle \sum m_i \chi^{(i)}, \sum m_j \chi^{(j)} \rangle = \sum_{i=1}^{k} m_i^2$

4) We just showed χ irreducible $\Rightarrow \langle \chi, \chi \rangle = 1$. Suppose $\langle \chi, \chi \rangle = 1 = \sum_{i=1}^{k} m_i^2$. Then $\exists!\, j \in \{1, \dots, k\}$ such that $m_j = 1$, $m_i = 0$ else, i.e. $\chi = \chi^{(j)}$ which is irreducible.

5) We know $X \cong Y \Rightarrow \chi = \psi$. Suppose $\chi = \psi$. Assume $Y \cong n_1 X^{(1)} \oplus \dots \oplus n_k X^{(k)}$, allowing for $n_i = 0$. $n_j = \langle \psi, \psi^{(j)} \rangle = \langle \chi, \chi^{(j)} \rangle = m_j$. Thus, $Y \cong m_1 X^{(1)} \oplus \dots \oplus m_k X^{(k)} \cong X$.

Prop: Let G be finite. Let $V^{(1)}, V^{(2)}, \dots$ be the complete list of inequivalent irreducible representations. Suppose $\mathbb{C}[G] \cong m_1 V^{(1)} \oplus m_2 V^{(2)} \oplus \dots$. Then $\dim V^{(i)} = m_i$ and $|G| = \sum_i (\dim V^{(i)})^2$.

Pf: Let $\chi = \chi^{reg}$ be the character of $\mathbb{C}[G]$, and $\chi^{(i)}$ be the character of $V^{(i)}$. Then

$$\chi = \sum_i m_i \chi^{(i)} \quad \text{and} \quad m_i = \langle \chi, \chi^{(i)} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi^{(i)}(g^{-1})$$

But

$$\chi(g) = \begin{cases} |G|, & g = e \\ 0, & g \neq e \end{cases} \implies m_i = \frac{1}{|G|} |G| \chi^{(i)}(e) = \chi^{(i)}(e) = \dim V^{(i)}$$

Thus, $m_i = \dim V^{(i)}$. Then, $|G| = \sum m_i^2 = \sum (\dim V^{(i)})^2$ as desired.

Lemma: Let G be finite. As vector spaces, $\mathbb{C}[G] \cong \operatorname{End}(\mathbb{C}[G])$.

Pf: Define $\varphi : \mathbb{C}[G] \to \operatorname{End}(\mathbb{C}[G])$ by $\varphi(\vec{v}) = \varphi_{\vec{v}} : \mathbb{C}[G] \to \mathbb{C}[G]$ where $\varphi_{\vec{v}}(\vec{w}) = \vec{w}\vec{v}$. We show φ is linear and bijective. Note that $\varphi(c\vec{v}_1 + d\vec{v}_2)(\vec{w}) = (c\varphi(\vec{v}_1) + d\varphi(\vec{v}_2))(\vec{w})$ so φ is linear. Suppose $\varphi(\vec{v}) = 0$, i.e. $\varphi(\vec{v})(\vec{w}) = 0$, $\forall \vec{w} \in \mathbb{C}[G]$. Then $\varphi(\vec{v})(\vec{e}) = \varphi_{\vec{v}}(\vec{e}) = \vec{e} \cdot \vec{v} = \vec{0}$. Thus, $\vec{v} = \vec{0}$. Hence, $\ker \varphi = \{0\}$ so φ is injective. Let $\Theta \in \operatorname{End}(\mathbb{C}[G])$. Then $\exists \vec{v} \in \mathbb{C}[G]$ s.t. $\Theta(\vec{e}) = \vec{v}$. Then $\forall g \in G$, $\Theta(\vec{g}) = \Theta(\vec{g} \cdot \vec{e}) = \vec{g}\,\Theta(\vec{e}) = \vec{g} \cdot \vec{v} = \varphi_{\vec{v}}(\vec{g})$. Thus, $\Theta(\vec{g}) = \varphi_{\vec{v}}(\vec{g})$, $\forall g \in G$. Thus, $\Theta = \varphi(\vec{v})$. Hence φ is surjective.

Def: Let A and B be algebras over $\mathbb{C}$. Then $\varphi : A \to B$ is an <u>anti-isomorphism</u> if φ is an isomorphism of vector spaces and $\varphi(a_1 a_2) = \varphi(a_2)\varphi(a_1)$.

Lemma: Let $G = \{g_1, \dots, g_n\}$. Then $\mathbb{C}[G]$ is anti-isomorphic to $\operatorname{End}(\mathbb{C}[G])$.

Pf: $\varphi : \mathbb{C}[G] \to \operatorname{End}(\mathbb{C}[G])$ is a vector space isomorphism. Also, $\forall \vec{v}_1, \vec{v}_2 \in \mathbb{C}[G]$,

$$\varphi(\vec{v}_1 \vec{v}_2)(\vec{w}) = \varphi_{\vec{v}_1 \vec{v}_2}(\vec{w}) = \vec{w}\,\vec{v}_1 \vec{v}_2 = \varphi_{\vec{v}_1}(\vec{w}) \cdot \vec{v}_2 = \varphi_{\vec{v}_2} \varphi_{\vec{v}_1}(\vec{w}) = \varphi(\vec{v}_2)\varphi(\vec{v}_1)(\vec{w})$$

<u>Thm</u>: Let G be finite and $\mathbb{C}[G] = \oplus m_i V^{(i)}$ be a decomposition into inequivalent irreducible representations. The number of $V^{(i)}$ is equal to the number of conjugacy classes of G.

<u>Pf</u>: We have seen that $\# V^{(i)} = \dim Z(\mathrm{End}(\mathbb{C}[G]))$. But $\mathrm{End}(\mathbb{C}[G])$ is anti-isomorphic to $\mathbb{C}[G]$. Thus $Z(\mathrm{End}(\mathbb{C}[G]))$ is anti-isomorphic to $Z(\mathbb{C}[G])$. Thus, $\dim Z(\mathrm{End}(\mathbb{C}[G])) = \dim Z(\mathbb{C}[G])$. Let $K_1, \dots, K_k$ be a complete list of distinct conjugacy classes in G. Define

$$z_i = \sum_{g \in K_i} \vec{g} \quad \in \mathbb{C}[G]$$

We show $\{z_1, \dots, z_k\}$ is a basis for $Z(\mathbb{C}[G])$. First, we show $z_i \in Z(\mathbb{C}[G])$. For $h \in G$, we have $\vec{h} z_i \vec{h^{-1}} = h(\sum_{g \in K_i} g) h^{-1} = \sum_{g \in K_i} hgh^{-1} = \sum_{\tilde{g} \in K_i} \tilde{g} = z_i$. Hence $hz_ih^{-1} = z_i$ and $hz_i = z_i h$, $\forall h \in G$ and extending linearly gives $z_i \in Z(\mathbb{C}[G])$. Suppose $\vec{v} = \sum_{i=1}^{n} c_i g_i \in Z(\mathbb{C}[G])$. Then $\vec{h}\vec{v} = \vec{v}\vec{h}$, $\forall h \in G$ so

$$\vec{v} = h\vec{v}h^{-1} = h\left(\sum_{i=1}^{n} c_i g_i\right) h^{-1} = \sum_{i=1}^{n} c_i h g_i h^{-1} = \sum_{i=1}^{n} c_i g_i$$

Thus, all elements of the same conjugacy class have the same coefficient c_j. So, combining like terms, $\vec{v} = \sum_{i=1}^{k} d_i z_i$ for some $d_i \in \mathbb{C}$.

<u>Cor</u>: The irreducible characters of G form an orthonormal basis of the space of class functions $R(G)$.

<u>Pf</u>: Irreducible characters are orthonormal and thus independent. They span $R(G)$ since there are as many irreducible characters as there are conjugacy classes of G, which is the dimension of $R(G)$.

<u>Thm</u>: (Character relations of the second kind) Let K and L be conjugacy classes of G. Then

$$\sum_{\chi \text{ irred.}} \chi_K \overline{\chi_L} = \frac{|G|}{|K|} \delta_{KL}$$

<u>Note</u>: This shows that the columns of the character table are also orthogonal.

<u>Pf</u>: Let $\lambda = \sqrt{|G|/|K|}$ and construct a new character table whose (i,j) entry is $\lambda \chi^{(i)}_{K_j}$. Note that character relations of the first kind state

$$\langle \chi^{(i)}, \chi^{(j)} \rangle = \frac{1}{|G|} \sum_K \chi^{(i)}_K \overline{\chi^{(j)}_K} |K| = \delta_{ij}$$

This doesn't imply orthonormality of the rows of the table. However, using arithmetic on $\lambda \chi_K$ we have $\delta_{ij} = \langle \lambda \chi^{(i)}, \lambda \chi^{(j)} \rangle$. Since this augmented matrix has orthonormal rows, it has orthonormal columns. Thus,

$$\sum \chi_K \overline{\chi_L} = \delta_{KL} \lambda^2$$

Ex: S_3. Let $\chi^{(1)}$ and $\chi^{(2)}$ correspond to the trivial and sign representations. We solve for $\chi^{(3)}$. Let $d = \deg \chi^{(3)}$. We have $|G| = \sum_{i=1}^{3} (\dim V_i)^2 = 1 + 1 + d^2$ so $d = 2$. Thus, $\chi^{(3)}(e) = 2$.

$$0 = \sum_{i=1}^{3} \chi^{(i)}(e)\, \overline{\chi^{(i)}}(12) = 1\cdot 1 + 1\cdot(-1) + 2\cdot \overline{\chi^{(3)}}(12) \Rightarrow \overline{\chi^{(3)}}(12) = 0$$

$$0 = \sum_{i=1}^{3} \chi^{(i)}(e)\, \overline{\chi^{(i)}}(123) = 1\cdot 1 + 1\cdot 1 + 2\cdot \overline{\chi^{(3)}}(123) \Rightarrow \overline{\chi^{(3)}}(123) = -1$$

So $\chi^{(3)} = (2, 0, 0, 0, -1, -1)$.

Prop: Let V be a G-module and W be an H-module. Then $V \otimes W$ is a $G \times H$-module with $(g,h)\cdot(\vec{v} \otimes \vec{w}) = (g\vec{v}) \otimes (h\vec{w})$ and extend linearly.

Def: Let $T: V \to V'$ be a linear transformation of vector spaces and $S: W \to W'$ be as well. Then $T \otimes S : V \otimes W \to V' \otimes W'$ is given by

$$(T \otimes S)(\vec{v} \otimes \vec{w}) = T(\vec{v}) \otimes S(\vec{w})$$

and is a linear transformation by extending linearly.

Prop: If $T: V \to V'$ is a G-homomorphism and $S: W \to W'$ is an H-homomorphism, then $T \otimes S$ is a $G \times H$-homomorphism.

Thm: Let $X: G \to GL_d$ and $Y: H \to GL_f$ be matrix representations. Then $X \otimes Y : G \times H \to GL_{df}$, $X \otimes Y(g,h) = X(g) \otimes Y(h)$ is also a matrix representation. Moreover, if X, Y, and $X \otimes Y$ have characters χ, ψ, and $\chi \otimes \psi$, respectively, then $\chi \otimes \psi(g,h) = \chi(g)\psi(h)$.

Pf: We show $X \otimes Y$ is a group homomorphism. Let $(g,h), (g',h') \in G \times H$.

$$\begin{aligned} X \otimes Y\big((g,h)\cdot(g',h')\big) &= X \otimes Y\big((gg', hh')\big) = X(gg') \otimes Y(hh') \\ &= X(g)X(g') \otimes Y(h)Y(h') = \big(X(g) \otimes Y(h)\big)\big(X(g') \otimes Y(h')\big) \\ &= X \otimes Y(g,h)\; X \otimes Y(g',h') \end{aligned}$$

so $X \otimes Y$ is a group homomorphism. Note for any matrices A and B,

$$\operatorname{Tr}(A \otimes B) = \operatorname{Tr}(a_{ij}B) = \sum_{i=1}^{d} a_{ii} \operatorname{Tr} B = \operatorname{Tr} A \operatorname{Tr} B$$

Thus, $\chi \otimes \psi(g,h) = \operatorname{Tr}(X(g) \otimes Y(h)) = \operatorname{Tr} X(g) \operatorname{Tr} Y(h) = \chi(g)\psi(h)$.

Thm: (1) If X and Y are irreducible representations of G and H, respectively, then $X \otimes Y$ is an irreducible representation of $G \times H$.

(2) If $X^{(1)}, \dots, X^{(k)}$ and $Y^{(1)}, \dots, Y^{(\ell)}$ are complete lists of inequivalent irreducible representations of G and H, respectively, then

$$\{ X^{(i)} \otimes Y^{(j)} \mid 1 \le i \le k,\ 1 \le j \le \ell \}$$

is a complete list of inequivalent irreducible representations of $G \times H$.

Pf: (1) We check that the inner product of the character with itself is 1.

$$\langle \chi\otimes\psi, \chi\otimes\psi\rangle = \frac{1}{|G\times H|}\sum_{(g,h)\in G\times H} \chi\otimes\psi(g,h)\,\chi\otimes\psi(g^{-1},h^{-1})$$
$$= \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{h\in H}\chi(g)\psi(h)\chi(g^{-1})\psi(h^{-1})$$
$$= \left(\frac{1}{|G|}\sum_{g\in G}\chi(g)\chi(g^{-1})\right)\left(\frac{1}{|H|}\sum_{h\in H}\psi(h)\psi(h^{-1})\right)$$
$$= 1\cdot 1 = 1$$

(2) We see that $\langle \chi^{(i)}\otimes\psi^{(j)}, \chi^{(m)}\otimes\psi^{(n)}\rangle = \delta_{im}\delta_{jn}$. Thus, $\chi^{(i)}\otimes\psi^{(j)}$ are pairwise inequivalent irreducible representations. Any complete list of inequivalent irreducible representations has length the number of conjugacy classes of $G\times H$ which is kl. This is the size of the set of inequivalent irreducible representations so it is a complete list.

Def: Let $H\leq G$ and X be a matrix representation of G. The <u>restriction</u> of X to H is defined by

$$X\downarrow^G_H(h) = X(h), \quad \forall h\in H$$

If X has character χ, then $X\downarrow^G_H$ has character $\chi\downarrow^G_H$.

Note: This is a matrix representation of H.

Remark: If V is a G-module, then V is an H-module.

Def: Let $H\leq G$, and $t_1,\dots,t_l$ be a transversal so $G = t_1H\sqcup t_2H\sqcup\dots\sqcup t_lH$. If Y is a representation of H, the induced representation given this transversal is defined by

$$Y\uparrow^G_H(g) = \left(Y(t_i^{-1}gt_j)\right) = \begin{pmatrix} Y(t_1^{-1}gt_1) & Y(t_1^{-1}gt_2) & \cdots & Y(t_1^{-1}gt_l)\\ \vdots & & \ddots & \vdots \\ Y(t_l^{-1}gt_1) & \cdots & \cdots & Y(t_l^{-1}gt_l)\end{pmatrix}$$

and $Y(g)=0$ for $g\notin H$.

Remark: If W is an H-module, $\mathrm{Ind}^G_H \cong \mathbb{C}[G]\otimes_{\mathbb{C}[H]} W$.

Note: $\mathrm{Ind}^G_H \cong \{f: G\to W \mid f(hg) = hf(g), \forall h\in H\}$

Ex: $G = S_3$, $H = \langle(2,3)\rangle$, $G = H\sqcup(12)H\sqcup(13)$. Let $Y=1$ be the trivial representation. Let $X = 1\uparrow^G_H$. Find $X(12)$.

$$X(12) = \begin{pmatrix} 1(e^{-1}(12)e) & 1(e^{-1}(12)(12)) & 1(e^{-1}(12)(13)) \\ 1((12)^{-1}(12)e) & 1((12)^{-1}(12)(12)) & 1((12)^{-1}(12)(13)) \\ 1((13)^{-1}(12)e) & 1((13)^{-1}(12)(12)) & 1((13)^{-1}(12)(13))\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}$$

Prop: $1\uparrow_H^G = \mathbb{C}\mathcal{H}$

Remark: The claim here is that in the basis $\{\vec{t_1H}, \dots, \vec{t_\ell H}\}$, the matrices for $\mathbb{C}\mathcal{H}$ and $1\uparrow_H^G$ are equal.

Pf: Let $X = 1\uparrow_H^G$ and let Z be the matrix representation of $\mathbb{C}\mathcal{H}$ in that basis. We show $X(g) = Z(g)$, $\forall g \in G$.

$$x_{ij}(g) = \begin{cases} 0, & t_i^{-1} g t_j \notin H \\ 1, & t_i^{-1} g t_j \in H \end{cases}$$

Also, $z_{ij}(g) \in \{0, 1\}$ because $Z(g)$ is a permutation matrix. Note that $x_{ij}(g) = 1 \iff 1\uparrow_H^G(t_i^{-1} g t_j) = 1 \iff t_i^{-1} g t_j \in H \iff g t_j \in t_i H \iff g(t_j H) = t_i H \iff z_{ij}(g) = 1$.
so $X(g) = Z(g)$, $\forall g \in G$ as desired.

Prop: $Y\uparrow_H^G$ is a representation of G.

Pf: We must show $Y\uparrow_H^G(g)$ is invertible and $Y\uparrow_H^G(gh) = Y\uparrow_H^G(g)\, Y\uparrow_H^G(h)$. To show invertibility, note $Y\uparrow_H^G(g)$ is a block permutation matrix, i.e. in each row and column $\exists!$ nonzero (block) entry $Y\uparrow_H^G(g) = (Y(t_i^{-1} g t_j))$ and $Y(t_i^{-1} g t_j) \neq 0$ iff $t_i^{-1} g t_j \in H \iff g t_j \in t_i H$. So in row j $\exists!$ i s.t. $g t_j \in t_i H$, as $\{t_1, \dots, t_\ell\}$ are a transversal for H. Computing $Y\uparrow_H^G(gh)$ completes the argument.

Prop: The induced representation is independent of the choice of transversal, up to equivalence of representations. In more detail, let $H \leq G$, and let Y be a matrix representation of H. Let $\{t_1, \dots, t_\ell\}$ and $\{s_1, \dots, s_\ell\}$ be two transversals of H giving rise to induced representations X and Z, respectively. Then X and Z are equivalent.

Pf: Let X, Y, Z have characters χ, ψ, φ. We show $\chi = \varphi$. We permute the transversal such that $t_i H = s_i H$ so $\exists h_i \in H$ s.t. $t_i = s_i h_i$. Then $t_i^{-1} g t_i \in H \iff h_i^{-1} s_i^{-1} g s_i h_i \in H$ $\iff s_i^{-1} g s_i \in H$. When both $t_i^{-1} g t_i, s_i^{-1} g s_i \in H$, they are conjugate in H. Then $\psi(t_i^{-1} g t_i) = \psi(s_i^{-1} g s_i)$ because ψ is a class function on H. Then we have

$$\chi(g) = \sum_{i=1}^{\ell} \operatorname{Tr} Y(t_i^{-1} g t_i) = \sum_{i=1}^{\ell} \psi(t_i^{-1} g t_i) = \sum_{i=1}^{\ell} \psi(s_i^{-1} g s_i) = \sum_{i=1}^{\ell} \operatorname{Tr} Y(s_i^{-1} g s_i) = \varphi(g).$$

Lemma: $\psi\uparrow_H^G(g) = \frac{1}{|H|} \sum_{x \in G} \psi(x^{-1} g x)$

Pf: $\psi\uparrow_H^G(g) = \sum_{i=1}^{\ell} \psi(t_i^{-1} g t_i)$. But, $\forall h \in H$, $\psi(g) = \psi(h^{-1} g h)$ so $\psi\uparrow_H^G(g) = \frac{1}{|H|} \sum_{i=1}^{\ell} \sum_{h \in H} \psi(h^{-1} t_i^{-1} g t_i h)$.
As h ranges in H and t_i goes from 1 to ℓ, $t_i h$ ranges over all of G as $G = t_1 H \sqcup \dots \sqcup t_\ell H$. Thus, $\psi\uparrow_H^G(g) = \frac{1}{|H|} \sum_{x \in G} \psi(x^{-1} g x)$

<u>Thm:</u> (Frobenius Reciprocity) Let $H \leq G$ and suppose χ and ψ are characters of G and H, respectively. Then,

$$\langle \psi\uparrow_H^G, \chi \rangle_G = \langle \psi, \chi\downarrow_H^G \rangle_H$$

<u>Pf:</u> $\langle \psi\uparrow_H^G, \chi\rangle = \frac{1}{|G|}\sum_{g\in G} \psi\uparrow_H^G(g)\chi(g^{-1}) = \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{x\in G}\psi(x^{-1}gx)\chi(g^{-1})$. We let $y = x^{-1}gx$ so $g^{-1} = xy^{-1}x^{-1}$ and have $= \frac{1}{|G|}\frac{1}{|H|}\sum_{g\in G}\sum_{y\in G}\psi(y)\chi(xy^{-1}x^{-1}) = \frac{1}{|H|}\sum_{y\in G}\psi(y)\chi(y^{-1})$
$= \frac{1}{|H|}\sum_{y\in H}\psi(y)\chi(y^{-1}) = \langle \psi, \chi\downarrow_H^G\rangle$ as desired.

<u>Notation:</u> If t is a tableau, $\{t\}$ is the corresponding tabloid

$$\{t\} = \{t' \text{ shape } \lambda \mid t' \sim t\}$$

<u>Ex:</u> $t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad \{t\} = \begin{array}{ccc}\hline 4 & 1 & 2 \\ \hline 3 & & \\ \cline{1-1}\end{array}$

<u>Note:</u> $|\{\lambda\text{-tabloids}\}| = \frac{n!}{\lambda!}$ where $\lambda! = \lambda_1!\lambda_2!\ldots\lambda_\ell!$

<u>Def:</u> Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ be the corresponding <u>Young subgroup</u> of S_n is

$$S_\lambda = S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \ldots \times S_{\{n-\lambda_\ell+1,\ldots,n\}}$$

<u>Note:</u> Let $\alpha_k = \{\lambda_1 + \ldots + \lambda_{k-1}+1, \ldots, \lambda_1 + \ldots + \lambda_k\}$. Then $S_\lambda = S_{\alpha_1} \times \ldots \times S_{\alpha_\ell}$.

<u>Ex:</u> $(3,1) \vdash 4$, $S_{(3,1)} = S_{\{1,2,3\}} \times S_{\{4\}}$

<u>Def:</u> Let $\pi_1, \ldots, \pi_k$ be a transversal for S_λ. Then $1\uparrow_{S_\lambda}^{S_n}$ is the matrix representation of

$$V^\lambda = \mathbb{C}\{\overrightarrow{\pi_1 S_\lambda}, \ldots, \overrightarrow{\pi_k S_\lambda}\}$$

where for $\sigma \in S_n$, $\sigma(\overrightarrow{\pi_i S_\lambda}) = \overrightarrow{(\sigma\pi_i)S_\lambda}$.

<u>Note:</u> S_n acts on the set of Young tabloids of shape $\lambda \vdash n$ by $\pi\{t\} = \{\pi t\}$.

<u>Def:</u> Let $\lambda \vdash n$. The <u>permutation module</u> associated to λ is

$$M^\lambda = \mathbb{C}\{\{t_1\}, \ldots, \{t_k\}\}$$

where $\{t_1\}, \ldots, \{t_k\}$ is a complete list of λ-tabloids.

<u>Ex:</u> $\lambda = (n)$, $\{t\} = \begin{array}{c}\hline 1\ 2\ \ldots\ n \\ \hline\end{array}$ is a complete list, $M^\lambda = \mathbb{C}\{\{t\}\}$. For $\pi \in S_n$, $\pi\{t\} = \{t\}$. This is the trivial representation.

<u>Ex:</u> $\lambda = (1^n) = (1, 1, \ldots, 1)$, each tableau of shape λ is its own equivalence class and there is a bijection between permutations and tableaus. Thus, M^λ is the regular representation of S_n.

Ex: $\lambda = (n-1, 1) \vdash n$, $\frac{\overline{a_1\ a_2\ \cdots\ a_{n-1}}}{\underline{a_n}} \longleftrightarrow \{a_n\}$, $M^\lambda \cong \mathbb{C}\{\vec{1}, \ldots, \vec{n}\}$ which is the defining representation.

Ex: Our three examples give the following table for S_3

S_3	$K_{(1^3)}$	$K_{(2,1)}$	$K_{(3)}$
$\varphi^{(3)}$	1	1	1
$\varphi^{(2,1)}$	3	1	0
$\varphi^{(1^3)}$	6	0	0

Def: A G-module M is cyclic if $\exists \vec{v} \in M$ s.t. $M = \mathbb{C}G\vec{v}$, where $G\vec{v} = \{g\vec{v} \mid g \in G\}$

Thm: Let $\lambda \vdash n$. The permutation module M^λ is cyclic, generated by any λ-tabloid of dimension $n!/\lambda!$.

Pf: $M^\lambda = \mathbb{C}\{\overrightarrow{\{t_1\}}, \ldots, \overrightarrow{\{t_k\}}\}$ where $\{\{t_i\}\}$ is a complete list of λ-tabloid so $\dim M^\lambda = n!/\lambda!$. Now let t and t' be λ-tabloids. Then they differ in their entries only so $\exists \pi$ s.t. $\pi(t) = t'$, $\pi(t_{ij}) = t'_{ij}$. Thus, $S_n \overrightarrow{\{t\}} = \{\overrightarrow{\{\pi t\}} \mid \pi \in S_n\} = \{\overrightarrow{\{t_1\}}, \ldots, \overrightarrow{\{t_k\}}\}$. Thus, $M^\lambda = \mathbb{C}S_n\overrightarrow{\{t\}}$

Thm: Let $\lambda \vdash n$. Then $V^\lambda \cong M^\lambda$ as S_n-modules.

Pf: We claim V^λ is cyclic. Indeed, $V^\lambda = \mathbb{C}S_n\overrightarrow{S_\lambda} = \mathbb{C}\{\overrightarrow{\pi_1 S_\lambda}, \ldots, \overrightarrow{\pi_k S_\lambda}\}$. Let $\Theta(\overrightarrow{\pi_i S_\lambda}) = \{\overrightarrow{\pi_i t_\lambda}\}$ and extend linearly. For S_n, $\Theta(\sigma\overrightarrow{\pi_i S_\lambda}) = \{\overrightarrow{\sigma\pi_i t_\lambda}\}$.

Def: Let $\lambda, \mu \vdash n$, $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, $\mu = (\mu_1, \ldots, \mu_m)$. Then λ dominates μ, denoted $\lambda \trianglerighteq \mu$, if $\lambda_1 + \ldots + \lambda_i \geq \mu_1 + \ldots + \mu_i$, $\forall i \geq 1$ ($\lambda_i = 0$ for $i > \ell$, $\mu_i = 0$ for $i > m$).

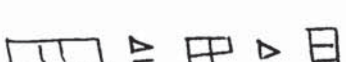
Remark: $(\{\lambda \vdash n\}, \trianglerighteq)$ is a poset.

Ex:  or $(3,2) \trianglerighteq (2,1^3)$

Def: Let $a, b \in (A, \leq)$. Then a is covered by b, denote $a \prec b$, if $a < b$ and $\nexists c \in A$ such that $a < c < b$.

Def: The Hasse diagram of the poset $(A, \leq)$ is the directed graph Γ defined by $V_\Gamma = A = \{a \mid a \in A\}$ and $E_\Gamma = \{(a,b) \mid a \prec b\}$.

Ex: $n = 3$

 $(3) \longleftarrow (2,1) \longleftarrow (1^3)$

Def: The <u>lexicographic</u> order on partitions is defined by $\lambda < \mu \iff \exists i$ s.t. $\lambda_j = \mu_j$, $j < i$ and $\lambda_i < \mu_i$.

Note: This is a total order.

Def: Let $\leq_1$ and $\leq_2$ be two partial orders on the set A. We say $\leq_2$ is a refinement of $\leq_1$ if $a \leq_1 b \Rightarrow a \leq_2 b$.

Prop: The lexicographical order is a refinement of the dominance order.

Pf: Let $\lambda, \mu \vdash n$. Suppose $\lambda \trianglerighteq \mu$. We show $\lambda \geq \mu$. Either $\lambda = \mu$ or $\lambda \triangleright \mu$. If $\lambda = \mu$ then $\lambda \geq \mu$. Otherwise, $\lambda \neq \mu$ and $\lambda \triangleright \mu$. Choose the smallest k s.t. $\lambda_k \neq \mu_k$. $\lambda \trianglerighteq \mu \Rightarrow$ for all i, $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j$. But $\sum_{j=1}^{k-1} \lambda_j = \sum_{j=1}^{k-1} \mu_j$ and $\sum_{j=1}^{k} \lambda_j > \sum_{j=1}^{k} \mu_j \Rightarrow \lambda_k > \mu_k$. Thus, $\lambda \geq \mu$.

Def: Let t be a tableau with row $R_1, \dots, R_k$ and columns $C_1, \dots, C_\ell$. The <u>row stabilizer</u> and <u>column stabilizer</u> of t are

$$R_t = S_{R_1} \times S_{R_2} \times \dots \times S_{R_k}$$
$$C_t = S_{C_1} \times S_{C_2} \times \dots \times S_{C_\ell}$$

Note: These are the subgroups of S_n fixing rows and columns of t, respectively.

Remark: $R_t \cong S_\lambda$, t shape λ.

Remark: $\{t\} = R_t t = \{\pi t \mid \pi \in R_t\}$

Notation: Let $H \subseteq S_n$ be a subset.

$$H^+ = \sum_{\pi \in H} \pi \in \mathbb{C}[S_n] \qquad H^- = \sum_{\pi \in H} \operatorname{sgn}(\pi)\, \pi \in \mathbb{C}[S_n]$$

Note: $\mathbb{C}[S_n] \circlearrowright M^\lambda$ so H^+ and H^- are not vectors.

Remark: $R_t^+ = \sum_{\pi \in R_t} \pi \in \mathbb{C}[S_n]$ which acts on $M^\lambda = \mathbb{C}\{\overrightarrow{\{t_1\}}, \dots, \overrightarrow{\{t_k\}}\}$.

Def: Let t be a tableau of shape λ. The <u>polytabloid</u> of t is

$$\vec{e_t} = \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\, \overrightarrow{\{\pi t\}} \in M^\lambda$$

Def: Let $\lambda \vdash n$. The <u>Specht module</u> S^λ is the submodule of M^λ spanned by the polytabloids of shape λ, i.e. $\vec{e_t}$, t shape λ.

__Def:__ Let G act on the set A. Let $B \subseteq A$, $H \leq G$.

(1) The __orbit__ of $a \in A$ under $H \leq G$ is $Ha = \{ha \mid h \in H\}$

(2) The __stabilizer__ of $b \in B$ is $stab(B) = \{g \in G \mid gB = B\}$, $stab(a) = stab(\{a\})$

(3) The action of G on A is __transitive__ if there is only one G-orbit in A, i.e. $\forall a \in A$, $Ga = A$, i.e. $\forall a, a' \in A$, $\exists g \in G$ s.t. $ga = a'$.

(4) The action of G on A is __free__ if $stab(a) = \{e\}$, $\forall a \in A$, i.e. $\forall a \in A$, $ga = a \Rightarrow g = e$.

__Def:__ Let $\rho: G \to GL(V)$ be a representation of G. Then ρ is a __faithful representation__ if ρ is injective.

__Def:__ Let V be an R-module. Let $U \subseteq V$. The __annihilator__ of U is

$$Ann(U) = \{r \in R \mid ru = 0, \forall u \in U\}$$

__Def:__ V is a __faithful R-module__ if $Ann(V) = \{0\}$.

__Note:__ If $r\vec{v} = s\vec{v}$, $\forall \vec{v} \in V$, $r \neq s$, then $(r-s)\vec{v} = 0$, $\forall \vec{v} \in V \Rightarrow V$ is not a faithful R-module.

__Remark:__ Let V be a G-module by linear extension, V is a $\mathbb{C}[G]$-module. If V is a faithful $\mathbb{C}[G]$-module, then $\rho: G \to GL(V)$, $\rho(g)\vec{v} = g\vec{v}$, is a faithful representation.

__Note:__ The converse is not true. Consider $\mathbb{C}[S_4]$ and the defining representation, $\rho: S_4 \to GL(V)$, $V = \mathbb{C}\{\vec{1}, \vec{2}, \vec{3}, \vec{4}\}$. This is faithful since each permutation has a unique permutation matrix. Consider $V = \mathbb{C}\{\vec{1}, \dots, \vec{4}\}$, $GL(V) \subseteq \mathbb{C}^{16}$, as a $\mathbb{C}[S_4]$-module, $\dim(\mathbb{C}[S_4]) = 24$. $Ann(V) = \{r = \sum c_i \pi_i \mid r\vec{i} = 0,\ \vec{i} = \vec{1}, \dots, \vec{4}\}$ so we want to find $\sum c_i \pi_i \vec{v} = 0$. Since we can't have 24 linearly independent vectors in 16 dimensions, the solution is trivial.

__Ex:__ Consider $t = \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$. We want to find $\vec{e_t}$.

$$\begin{aligned}\vec{e_t} &= sgn(e)\overrightarrow{\{et\}} + sgn((34))\overrightarrow{\{(34)t\}} \\ &= \overrightarrow{\{et\}} - \overrightarrow{\{(34)t\}} = \begin{array}{ccc}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{ccc}\hline 3 & 2 & 1 \\ \hline 4 \\ \cline{1-1}\end{array}\end{aligned}$$

__Notation:__ Let $K_t = C_t^-$ where $C_t^- = \sum_{\pi \in C_t} sgn(\pi)\pi$.

__Remark:__ $C_t^-\{t\} = \vec{e_t}$

__Prop:__ $K_t = K_{C_1} K_{C_2} \dots K_{C_\ell}$

__Note:__ C_i is a tableau. $K_{C_i} = \sum_{\pi \in C_{C_i}} sgn(\pi)\pi = \sum_{\pi \in S_{C_i}} sgn(\pi)\pi$. $C_i = \begin{array}{|c|}\hline C_{1i} \\ \hline \vdots \\ \hline C_{k_i i} \\ \hline\end{array}$.

__Pf:__ Induction. Let $H_1, H_2 \leq S_n$ with disjoint support. ($supp(\pi) = \{i \in \{1, \dots, n\} \mid \pi(i) \neq i\}$)

$$\sum_{\pi \in H_1 H_2} sgn(\pi)\pi = \sum_{\sigma \in H_1} sgn(\sigma)\sigma \cdot \sum_{\tau \in H_2} sgn(\tau)\tau$$

__Ex:__ $t = \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$, $\vec{e_t} = K_t\overrightarrow{\{t\}}$, $K_t = K_{C_1}K_{C_2}K_{C_3} = (e - (34))(e)(e) = e - (34)$

$K_t\{t\} = (e - (34))\overrightarrow{\{t\}} = \begin{array}{ccc}\hline 4 & 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} - \begin{array}{ccc}\hline 3 & 2 & 1 \\ \hline 4 \\ \cline{1-1}\end{array}$

Ex: $\lambda = (n) = \boxed{1 \cdots 1}$, $\{t\} = \overline{1\ 2 \ldots n}$. This is the unique tabloid of shape λ.
$C_t = \{e\}$, $\vec{e}_t = \sum_{\pi \in C_t} \mathrm{sgn}(\pi)\pi\overrightarrow{\{t\}} = \overrightarrow{\{t\}}$. Thus, $S^{(n)} = \mathbb{C}\{t\}$ and $\pi\vec{e}_t = \vec{e}_{\pi t} = \vec{e}_t$. Thus, $S^{(n)}$ is the trivial representation on S_n.

Recall: Let $\sigma, \pi \in S_n$. Suppose $\pi = (a_1\ a_2 \ldots a_k)(b_1\ b_2 \ldots b_l)\ldots$. Then

$$\sigma\pi\sigma^{-1} = (\sigma(a_1)\ \sigma(a_2) \ldots \sigma(a_k))(\sigma(b_1)\ \sigma(b_2) \ldots \sigma(b_l))\ldots$$

Pf: We show LHS = RHS. Choose $m \in \{1, \ldots, n\}$. Then, as $\sigma \in S_n$, σ is bijective and in particular surjective. Wlog $\sigma(a_1) = m$. Consider the LHS. $\sigma\pi\sigma^{-1}(m) = \sigma\pi\sigma^{-1}(\sigma(a_1)) = \sigma\pi(a_1) = \sigma(a_2)$. Consider the RHS. $m = \sigma(a_1) \mapsto \sigma(a_2)$.

Remark: The action of S_n on M^λ is given by conjugation.

Ex: $(1\,3) \cdot \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array} \implies (1\,3)\cdot(1\,2)(3) = (3\,2)(1)$

Lemma: Let $\lambda \vdash n$, shape $t = \lambda$, $\pi \in S_n$.

(1) $R_{\pi t} = \pi R_t \pi^{-1}$
(2) $C_{\pi t} = \pi C_t \pi^{-1}$
(3) $K_{\pi t} = \pi K_t \pi^{-1}$
(4) $\vec{e}_{\pi t} = \pi \vec{e}_t$

Pf: (1) $\sigma \in R_{\pi t} \iff \sigma\{\pi t\} = \{\pi t\} \iff \sigma\pi\{t\} = \pi\{t\} \iff \pi^{-1}\sigma\pi\{t\} = \{t\} \iff \pi^{-1}\sigma\pi \in R_t \Leftrightarrow \sigma \in \pi R_t \pi^{-1}$
(2) and (3) are similar. (4) $\vec{e}_{\pi t} \Leftrightarrow K_{\pi t}\overrightarrow{\{\pi t\}} \Leftrightarrow \pi K_t \pi^{-1}\pi\{t\} \Leftrightarrow \pi K_t \overrightarrow{\{t\}} \Leftrightarrow \pi\vec{e}_t$.

Prop: Specht modules S^λ are cyclic and generated by any polytabloid of shape λ.

Pf: We have $S^\lambda = \mathbb{C}\{\vec{e}_t \mid t \text{ shape } \lambda\} = \mathbb{C}\{\vec{e}_{\pi t} \mid \pi \in S_n\} = \mathbb{C}\{\pi\vec{e}_t \mid \pi \in S_n\} = \mathbb{C}S_n\{\vec{e}_t\}$, t shape λ.

Ex: $\lambda = (1^n) \vdash n$. Find $S^\lambda \subseteq M^\lambda$. Choose t of shape λ. $t = \begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n \\ \hline\end{array}$. $C_t = S_n$. $\vec{e}_t = \sum_{\sigma \in S_n}(\mathrm{sgn}\,\sigma)\sigma\overrightarrow{\{t\}}$ so
$\vec{e}_{\pi t} = \pi\vec{e}_t = \pi\left(\sum_{\sigma \in S_n}(\mathrm{sgn}\,\sigma)\sigma\overrightarrow{\{t\}}\right) = \sum_{\sigma \in S_n}(\mathrm{sgn}\,\sigma)\pi\sigma\overrightarrow{\{t\}} = \sum_{\tau \in S_n}(\mathrm{sgn}\,\pi^{-1}\tau)\tau\overrightarrow{\{t\}} = (\mathrm{sgn}\,\pi)\sum_{\tau \in S_n}(\mathrm{sgn}\,\tau)\tau\overrightarrow{\{t\}} = (\mathrm{sgn}\,\pi)\vec{e}_t$.
Thus, $\vec{e}_{\pi t} = (\mathrm{sgn}\,\pi)\vec{e}_t = \pm\vec{e}_t$ and $S^\lambda = \mathbb{C}S_n\{\vec{e}_t\} = \mathbb{C}\{\vec{e}_t\}$ so S^λ is one-dimensional and $\pi\vec{e}_t = (\mathrm{sgn}\,\pi)\vec{e}_t$. Thus, $S^{(1^n)}$ is the sign representation.

Note: Let E be the sign representation. S^λ is the unique irreducible S_n-module such that $1_{S_\lambda} \subseteq S^\lambda\big\downarrow_{S_\lambda}^{S_n}$, $E_{S_{\lambda^*}} \subseteq S^\lambda\big\downarrow_{S_{\lambda^*}}^{S_n}$ where $\lambda^* = \lambda^T$, $t_{ij} \leftrightarrow t_{ji}$, is the conjugate of λ. From the perspective of induction, $M^\lambda = 1\big\uparrow_{S_\lambda}^{S_n}$, $E\big\uparrow_{S_\lambda}^{S_n} = M^{\lambda^*} \otimes E_{S_n}$, $\dim(\mathrm{Hom}_{S_n}(M^\lambda, M^{\lambda^*} \otimes E)) = 1$. $S^\lambda = \text{image } \alpha : M^\lambda \to M^\lambda \otimes E$, $\alpha \neq 0$.

Lemma: (Sign Lemma) Let $H \leq S_n$.

(1) If $\pi \in H$, $\pi H^- = H^-\pi = (\mathrm{sgn}\,\pi)H^-$, i.e. $\pi^- H^- = H^-$.
(2) $\forall \vec{u}, \vec{v} \in M^\lambda$, $\langle H^-\vec{u}, \vec{v}\rangle = \langle \vec{u}, H^-\vec{v}\rangle$
(3) $(a\,b) \in H \leq S_n$, then $H^- = c(e - (a\,b))$, $c \in \mathbb{C}[S_n]$
(4) If a, b are in the same row of tableau t, and $(a\,b) \in H$, then $H^-\overrightarrow{\{t\}} = \vec{0}$.

Prop: Let t be a tableau of shape λ. The action of $C_{\{t\}}$ on the set of λ-tabloids is transitive. If $\{s\}$ and $\{t\}$ are shape λ, then $\exists \pi \in C_{\{t\}}$ s.t. $\{s\} = \pi\{t\}$.

Lemma: (Dominance Lemma for Partitions) Let t^λ and s^μ be tableau of shape λ and μ. If for each i, the elements of row i of s are in different columns of t, then $\lambda \trianglerighteq \mu$.

Pf: We can sort the entries in each column of t^λ so that all of the entries of rows $1, 2, \ldots, i$ of S^μ all appear in the first i rows of t^λ. All entries of row 1 of s are in different columns of t. So # columns of $t \geq$ # columns of s and $\lambda_1 \geq \mu_1$. So we can sort the elements of t so that elements of row 1 of s appear in row 1 of t. We then repeat. Thus, $\lambda_1 + \ldots + \lambda_i =$ # elements in rows 1 through i of $t \geq$ # elements in rows 1 to i of $s = \mu_1 + \ldots + \mu_i$.

Lemma: (Sign Lemma) Let $H \leq S_n$, $\pi \in H$, $\pi^- = (\operatorname{sgn} \pi)\pi$.

(1) $\pi H^- = H^- \pi = (\operatorname{sgn}\pi) H^-$, i.e. $\pi^- H^- = H^-$

(2) $\forall \vec{u}, \vec{v} \in M^\lambda$, $\langle H^- \vec{u}, \vec{v} \rangle = \langle \vec{u}, H^- \vec{v} \rangle$

(3) $(a\ b) \in H \Rightarrow H^- = c(e - (a\ b))$ for some $c \in \mathbb{C}[S_n]$

(4) If a, b are in the same row of tableau t, and $(a\ b) \in H$, then $H^- \overrightarrow{\{t\}} = \vec{0}$.

Pf: (1) $\pi H^- = \pi \left(\sum_{\sigma \in H} (\operatorname{sgn}\sigma)\sigma \right) = \sum_{\sigma \in H} (\operatorname{sgn}\sigma)\pi\sigma = \sum_{\sigma \in H} (\operatorname{sgn}\pi^{-1})(\operatorname{sgn}\pi\sigma)\pi\sigma = (\operatorname{sgn}\pi) \sum_{\pi\sigma \in H} (\operatorname{sgn}\pi\sigma)\pi\sigma = (\operatorname{sgn}\pi) H^-$

(2) $\langle H^- \vec{u}, \vec{v} \rangle = \left\langle \sum_{\sigma \in H} (\operatorname{sgn}\sigma)\sigma\vec{u}, \vec{v} \right\rangle = \sum_{\sigma \in H} \langle (\operatorname{sgn}\sigma)\sigma\vec{u}, \vec{v} \rangle = \sum_{\sigma \in H} \langle \vec{u}, (\operatorname{sgn}\sigma^{-1})\sigma^{-1}\vec{v} \rangle = \left\langle \vec{u}, \sum_{\sigma^{-1} \in H} (\operatorname{sgn}\sigma^{-1})\sigma^{-1}\vec{v} \right\rangle = \langle \vec{u}, H^- \vec{v} \rangle$

(3) Let $K = \langle (a\ b) \rangle = \{e, (a\ b)\} \leq H$. Choose a transversal $\{\pi_i\}$ of K in H, $H = \bigsqcup_i \pi_i K$.
Then, $H^- = \left(\sum_i \pi_i^- \right) K^- = \left(\sum_i \pi_i^- \right)(e - (a\ b))$.

(4) $H^- \overrightarrow{\{t\}} = c(e - (a\ b))\overrightarrow{\{t\}} = c(\overrightarrow{\{t\}} - \overrightarrow{\{t\}}) = \vec{0}$

Cor: Let $\lambda, \mu \vdash n$. Let t, s have shape λ, μ, respectively. If $K_t \overrightarrow{\{s\}} \neq \vec{0}$, then $\lambda \trianglerighteq \mu$. If $\lambda = \mu$, $K_t \overrightarrow{\{s\}} = \pm K_t \overrightarrow{\{t\}} = \pm \vec{e}_t$.

Pf: First, we show any two elements in the same row of S^μ must be in different columns of t^λ. If a and b are in the same row of S^μ, and further they are in the same column of t, then $(a\ b) \in C_t$. Then $K_t = C_t^- = c(e - (a\ b))$ by (3) and $K_t \overrightarrow{\{s\}} = \vec{0}$ by (4). Thus, a and b are not in the same column. Hence, $\lambda \trianglerighteq \mu$ by the Dominance Lemma. Now suppose $\lambda = \mu$. $\exists \pi \in C_t$ s.t. $\{s\} = \pi\{t\}$.
Then, $K_t \overrightarrow{\{s\}} = K_t \overrightarrow{\{\pi t\}} = K_t \pi \overrightarrow{\{t\}} = (\operatorname{sgn}\pi) K_t \overrightarrow{\{t\}} = \pm \vec{e}_t$.

Thm: (Kernel Intersection Theorem) $S^\lambda = \bigcap_\varphi \ker \varphi$ where the intersection is over all S_n-module homomorphisms $\varphi: M^\lambda \to M^\mu$ where $\lambda \lhd \mu$.

Note: There are numbers called Kostka numbers which count the multiplicity of irreducibles in $M^\lambda \cong \bigoplus_{\mu: \lambda \unlhd \mu} K_{\mu\lambda} S^\mu$. It turns out $K_{\mu\lambda} \neq 0$ if and only if $\lambda \unlhd \mu$.

Thm: (Wedderburn) Let G be a finite group. Then $\mathbb{C}[G]$ is isomorphic to a direct sum of matrix rings over $\mathbb{C}$.

$$\mathbb{C}[G] \cong \mathbb{C}^{d_1 \times d_1} \oplus \ldots \oplus \mathbb{C}^{d_k \times d_k}$$

Furthermore, the d_i's are unique up to permutation. Any isomorphism $D: \mathbb{C}[G] \to \bigoplus_{i=1}^k \mathbb{C}^{d_i \times d_i}$ is called a discrete Fourier transform. If $D = D_1 \oplus \ldots \oplus D_k$, then the D_i form a complete set of irreducible representations for G.

Ex: If $G = S_3$, then $\mathbb{C}[S_3] \cong \mathbb{C}^{1\times1} \oplus \mathbb{C}^{2\times2} \oplus \mathbb{C}^{1\times1}$.

Ex: If $G = S_2$, then $\mathbb{C}[S_2] \cong \mathbb{C}^{1\times1} \oplus \mathbb{C}^{1\times1}$. $D_1: e \mapsto [1], (12) \mapsto [1]$, $D_2: e \mapsto [1], (12) \mapsto [-1]$

Cor: Let shape$(t) = \lambda$, $\vec{v} \in M^\lambda$. Then $K_t \vec{v}$ is a multiple of $\vec{e_t}$.

Pf: Let $\overrightarrow{\{t_1\}}, \ldots, \overrightarrow{\{t_k\}}$ be the distinct λ-tabloids. Then $\vec{v} = \sum_{i=1}^k c_i \overrightarrow{\{t_i\}}$ for scalars $c_i \in \mathbb{C}$. Thus $K_t \vec{v} = K_t \left(\sum c_i \overrightarrow{\{t_i\}}\right) = \sum c_i K_t \overrightarrow{\{t_i\}} = \sum c_i (\pm \vec{e_t}) = \left(\sum \pm c_i\right) \vec{e_t}$.

Thm: (Submodule Theorem) Let $U \subseteq M^\lambda$ be a submodule. Then $S^\lambda \subseteq U$ or $U \subseteq (S^\lambda)^\perp$.

Pf: Let $\vec{u} \in U$, shape$(t) = \lambda$. Then $\exists c \in \mathbb{C}$ such that $K_t \vec{u} = c\vec{e_t}$. Suppose $c \neq 0$, i.e. $\exists t, \vec{u} \in U$ such that $K_t \vec{u} \neq 0$. Note $\vec{u} \in U$, U submodule $\Rightarrow K_t \vec{u} \in U$. Thus $\vec{0} \neq c\vec{e_t} \in U \Rightarrow \vec{e_t} \in U$. But S^λ is cyclic, generated by $\vec{e_t}$, i.e. $S^\lambda = \mathbb{C}S_n \vec{e_t}$. Thus $S^\lambda \subseteq U$. Suppose otherwise that $K_t \vec{u} = \vec{0}$, $\forall \vec{u} \in U$, t shape λ. $\langle \vec{u}, \vec{e_t} \rangle = \langle \vec{u}, K_t \overrightarrow{\{t\}} \rangle = \langle K_t \vec{u}, \overrightarrow{\{t\}} \rangle = \langle 0, \overrightarrow{\{t\}} \rangle = 0$. But $\vec{e_t}$ spans S^λ. Thus, $\vec{u} \in (S^\lambda)^\perp$.

Cor: The Specht module S^λ is irreducible over $\mathbb{C}$.

Pf: There are no nontrivial submodules by the Submodule Theorem.

Prop: Let $\theta \in \mathrm{Hom}_{S_n}(S^\lambda, M^\mu)$. If θ is nonzero, then $\lambda \unrhd \mu$. If $\lambda = \mu$, $\exists c \in \mathbb{C}$ such that $\theta(\vec{v}) = c\vec{v}$, $\forall \vec{v} \in S^\lambda$.

Pf: Note $M^\lambda = S^\lambda \oplus (S^\lambda)^\perp$ so we may extend θ to $\hat{\theta}: M^\lambda \to M^\mu$ by $\hat{\theta}(\vec{v}) = \begin{cases} \theta(v), & \vec{v} \in S^\lambda \\ 0, & \vec{v} \in (S^\lambda)^\perp \end{cases}$

If θ is nonzero, $\exists$ basis vector $\vec{e_t}$ such that $\hat{\theta}(\vec{e_t}) \neq \vec{0}$. Then $\vec{0} \neq \hat{\theta}(\vec{e_t}) = \hat{\theta}(K_t \overrightarrow{\{t\}}) = K_t \hat{\theta}(\overrightarrow{\{t\}}) = K_t\left(\sum c_i \{s_i\}\right)$ for $\{s_i\} \in M^\mu$. So $\exists s_i$ such that $K_t \overrightarrow{\{s_i\}} \neq 0$ $\Rightarrow \lambda \unrhd \mu$. Suppose $\lambda = \mu$. Then $\theta(\vec{e_t}) = \theta(K_t \overrightarrow{\{t\}}) = K_t(\theta \overrightarrow{\{t\}}) = K_t \vec{v}$, $\vec{v} \in M^\lambda$. So by a corollary, $\exists c \in \mathbb{C}$ s.t. $K_t \vec{v} = c\vec{e_t}$. Then for any $\pi \in S_n$, $\theta(\vec{e}_{\pi t}) = \theta(\pi \vec{e_t}) = \pi \theta(\vec{e_t}) = \pi c \vec{e_t} = c \pi \vec{e_t} = c\vec{e}_{\pi t}$.

Thm: The set $\{S^\lambda \mid \lambda \vdash n\}$ is a complete set of irreducible S_n-modules over $\mathbb{C}$.

Pf: By the Submodule Theorem, S^λ is irreducible. The number of Specht modules is the number of inequivalent irreducible representations of S_n so the result follows if the Specht modules are pairwise inequivalent. If $S^\lambda \cong S^\mu$, then $\exists\, \theta \in \mathrm{Hom}(S^\lambda, M^\mu)$ that is nonzero as $S^\mu \subseteq M^\mu$. $\theta : S^\lambda \xrightarrow{\cong} S^\mu \hookrightarrow M^\mu$. and $\lambda \trianglerighteq \mu$. But the same argument shows $\mu \trianglerighteq \lambda$ so $\lambda = \mu$. Thus, if $\lambda \neq \mu$ then $S^\lambda \not\cong S^\mu$.